Hypothetical measures

W. D. Brinda and Elena Khusainova

A central aspect of measure theory is the extension of non-negative countably additive set functions (known as *premeasures*) to larger domains. A σ-finite premeasure μ defined on a semi-algebra has a *unique* extension to a measure on the σ-algebra generated by its domain, by the Carathéodory construction [Bogachev, 2007, Prop 1.3.10].[1] An interpretation of this in terms of real-world modeling is that the premeasure is a state of knowledge of a substance's mass on certain sets; the substance's mass on some other sets can be inferred by the nature of *mass*, that is, non-negativity, countable additivity, and the assumption that emptiness has zero mass. What about a set A that is not in the completion of the σ-algebra generated by the original domain? We may not be able to infer a mass that it *must* have, but we can still exclude some values. Any value strictly less than its μ-induced inner measure (supremum of masses of its subsets) should be considered unreasonable, as should any value strictly larger than its outer measure (infimum of masses of its supersets). In fact, if the outer measure of A is finite, then given any value z between the inner and outer measure of A there exists an extension of μ to a measure on the σ-algebra generated by the original domain and A that assigns a measure of z to A [Bogachev, 2007, Thm 1.12.14]. It is sensible to conclude that any value in that range might be the "true" mass of A. This reasoning seems preferable to an insistence that conditions must be imposed to avoid the possibility of "unmeasurable" sets.

This line of thinking can be implemented in a simpler and more powerful way than one might expect. Let μ be a premeasure on Ω with domain Σ, and let f be a function on Ω. Suppose $\mathcal{A} \subseteq 2^\Omega$ is at least fine enough that f is $\sigma(\Sigma \cup \mathcal{A})$-measurable. We define the *hypothetical measure* (we will say *hypomeasure*, for short) $\mu^{\mathcal{A}} f$ to be an indexed family where the indices are extensions of μ to $\sigma(\Sigma \cup \mathcal{A})$ and each such extension indexes the integral of f according to that measure.[2] One might prefer to omit the superscript by letting $\mathcal{A}$ be σ-algebra generated by f by default, or more generally letting $\mathcal{A}$ be the union of the σ-algebras generated by the functions that are to be integrated in the statement at hand. It is appropriate that this results in the hypomeasure notation being indistinguishable from the ordinary integral, because the hypomeasure extends the concept of integral: when f is Σ-measurable, the hypomeasure $\mu^{\sigma(f)} f$ is constant (equal to its ordinary integral's value) and can be treated as such.

[1] A measure has a unique extension to its completion, also via Carathéodory construction. And integration is a unique extension of the domain from indicator functions to measurable functions [Pollard, 2002, Sec 2.3].

[2] Our notation is intended to resemble (and extend) the de Finetti notation for integrals.

The facts and operations that are valid for ordinary measures continue to hold point-wise for hypomeasures. By proceeding *as if* every function were measurable, one produces equations and inequalities that hold *point-wise*, where the points are the extensions of μ.

Somewhere, include a sentence that points out the analogy to conditional probabilities and sub-σ-algebras. (in a footnote?)

To see why this approach is more powerful than just calculating inner and outer measures, consider the following simple example. Let f_1 be the indicator function of an unmeasurable set A, and let f_2 be 2 times the indicator function of A. Suppose A has inner measure 0 and outer measure 1. Then one cannot compare the integrals of f_1 and f_2 by comparing by their inner/outer measure ranges, which are $[0, 1]$ and $[0, 2]$ respectively. However, the hypomeasures approach allows us to unhesitatingly assert that the "integral" of f_1 is no greater than the "integral" of f_2, regardless of what the masses of currently unknown sets turn out to be.

The hypomeasures approach greatly simplifies our work by allowing us to treat every function as measurable.[3] *Measurability only becomes relevant to follow-up questions* regarding the range of a hypomeasure. If the σ-algebra generated by f is a subset of the μ-completion of the domain of μ, then the μ-hypomeasure of f is constant, being everywhere equal to the ordinary integral of f according to the completion of μ. Measurability of f by the completion of μ is necessary and sufficient for this constancy when μ is a finite measure [Halmos, 1974, Thm 14.F]. and what about an infinite measure?

Another follow-up question is perhaps concerning: do there exist *any* extensions of μ that can measure f? If so, we will call f *compatible* with μ. An extension to a finite or disjoint collection of sets is guaranteed to exist; see Proposition 1 below. However, incompatibility is possible; indeed, assuming Zorn's Lemma, no atomless measure can exist on a power set [Troitskii, 1994, Theorem 5]. Even a countable collection of sets has been devised that is incompatible with Lebesgue measure, assuming the continuum hypothesis [Bogachev, 2007, Cor 3.10.3].[4] However, such pathological functions are unusual in practice, so we suggest that a "presumption of innocence" is sensible. Furthermore, realize that there is nothing mathematically illegitimate about incompatible cases; they produce identities and inequalities that are vacuously true *point-wise* as there are not any points to check.[5]

[3] The hypomeasure idea is fairly straight-forward and has likely been discussed before somewhere.

[4] To clarify, the Vitali sets *are not* the example that we are referring to. It is easy to extend Lebesgue measure to include the Vitali sets, as they are disjoint [Bogachev, 2007, Thm 1.12.5]. The significance of the Vitali sets was that they demonstrated that there is no *translation-invariant* extension.

[5] Careful not to be misled, though. Consider [Mattner, 1999, Sec 2.2] in which a nonnegative integrand produces different results depending on the order of integration. Recall that Tonelli's Theorem requires product-measurability, which fails in Mattner's example. We can conclude that there is no extension of the measure for which the integrand is measurable; otherwise, Tonelli would apply and the iterated integrals would be valid. Thus, this is a case in which the hypomeasure has an empty domain.

When convenient, one can instruct the reader to *interpret "integrals" as hypomeasures*. In this way, identities and inequalities proven are mathematically legitimate regardless of measurability, and they are also realistically meaningful except in the pathological cases of incompatibility.

Proposition 1 Let $\mathcal{S} \subseteq 2^\Omega$ be a semi-ring of subsets of Ω, and let $\mu : S \to \mathbb{R}^+$ be countably additive with $\mu\,\emptyset = 0$. Let $\mathcal{A}$ be a collection of subsets of Ω. If all but finitely many of the sets in $\mathcal{A}$ are disjoint, then there exists an extension of μ to a measure with domain $\sigma(\mathcal{S} \cup \mathcal{A})$.

Proof. I'm pretty sure the statement is true. Much of it is standard material from probability theory that we just need to point to sources for. And the existence of extension to $\mathcal{A}$ for arbitrary measures seem like they might follow easily from the extension results for finite measures stated in Bogachev. The more interesting question is whether we can weaken the semi-ring part. A semi-ring is just the right amount of structure for which there exists a unique extension to a σ-algebra. If you just want to ensure the existence of *some* extension, can you use an even weaker type of collection? If it turns out that there is a weaker type of collection that is guaranteed to have an extension to a σ-algebra, then the result should be broken up into two propositions. □

References

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