Product Spaces

In Section REF, we saw a simple way to put measures together (eq:product-measure) to create a *product measure* on a product space. Here we will see another way to construct a measure on a product space from measures on the individual measure spaces. Essential to this discussion is the idea that an integral with respect to a measure on a product space can often be found by using an iterative integration procedure. We will also take another look at marginal measures. Finally, we will see that it is often advantageous to "rearrange" an ordinary measure space into a product space.

1 Constructing a measure on a product space

Let $(\mathbb{X}, \mathbb{A})$ and $(\mathbb{Y}, \mathbb{B})$ be measurable spaces. The ideas in this section extend to any finite number of measurable spaces in a straight-forward way. Extension to countably many is more subtle and is discussed (REF a section further down). Suppose $\Phi := \{\Phi_x : x \in \mathbb{X}\}$ is a *family of measures* on $(\mathbb{Y}, \mathbb{B})$. We say that Φ is a kernel from $(\mathbb{X}, \mathbb{A})$ to $(\mathbb{Y}, \mathbb{B})$ if for every $B \in \mathbb{B}$, the mapping $x \mapsto \Phi_x B$ is $\mathbb{A}$-measurable.

A kernel is called *σ-finite* if (DEFINE - make it intuitive if possible - see Pollard section 4.3 for definition).

1.1 True or false: if every measure in a kernel is a σ-finite measure, then that kernel is a σ-finite kernel.

1.2 Give an example of a σ-finite kernel whose measures are not σ-finite.

Theorem 1.1 (see Pollard, 2002, Theorems 4.20, 4.22)**.** *Let μ be a measure on $(\mathbb{X}, \mathbb{A})$ and Φ be a σ-finite kernel from $(\mathbb{X}, \mathbb{A})$ to $(\mathbb{Y}, \mathbb{B})$. Given any $f \in M^+(\mathbb{A} \otimes \mathbb{B})$,*

(i) $x \mapsto \int_{\mathbb{Y}} f(x, y) d\Phi_x(y)$ is $\mathbb{A}$-measurable,

(ii) the repeated integral $\int_{\mathbb{X}} \int_{\mathbb{Y}} f(x, y) d\Phi_x(y) d\mu(x) =: (\mu \otimes \Phi) f$ defines a measure on $(\mathbb{X} \times \mathbb{Y}, \mathbb{A} \otimes \mathbb{B})$.

1.3 Suppose a measure μ on $(\mathbb{X}_1 \times \mathbb{X}_2, \mathbb{A}_1 \otimes \mathbb{A}_2)$ is the product of μ_1 and μ_2, with μ_2 σ-finite. What measurability property of $f \in M^+(\mathbb{A}_1 \otimes \mathbb{A}_2)$ follows from Theorem 1.1?

The ability to find $(\mu \otimes \Phi)$-integrals by evaluating iterated integrals is extraordinarily convenient. The next two theorems provide us with additional conditions allowing the use of iterated integrals to find a *product measure's* integral (recall Section REF).

Theorem 1.2 (Tonelli's Theorem, Mukherjea, 1972, Theorem 1)**.** *Let $(\mathbb{X}, \mathbb{A}, \mu)$ be σ-finite and $(\mathbb{Y}, \mathbb{B}, \gamma)$ be semi-finite. If the product measure $\mu \otimes \gamma$ is semi-finite on $(\mathbb{X} \times \mathbb{Y}, \mathbb{A} \otimes \mathbb{B})$, then for any $f \in M^+(\mathbb{A} \otimes \mathbb{B})$,*

$$(\mu \otimes \gamma) f = \int_{\mathbb{X}} \int_{\mathbb{Y}} f(x, y) d\gamma(y) d\mu(x) = \int_{\mathbb{Y}} \int_{\mathbb{X}} f(x, y) d\mu(x) d\gamma(y).$$

Theorem 1.3 (Fubini's Theorem, see Mukherjea, 1972)**.** *Let $(\mathbb{X}, \mathbb{A}, \gamma)$ and $(\mathbb{Y}, \mathbb{B}, \mu)$ be measure spaces and $f \in M(\mathbb{A} \otimes \mathbb{B})$. If $(\gamma \otimes \mu)|f| < \infty$, then*

$$(\gamma \otimes \mu) f = \int_X \int_Y f(x, y) d\mu(y) d\gamma(x) = \int_Y \int_X f(x, y) d\gamma(x) d\mu(y).$$

Suppose $g \in M(\mathbb{A} \otimes \mathbb{B})$. In order to use Fubini's Theorem, one needs to check that g is integrable. Tonelli's Theorem is a perfect tool for the job, as it allows us to consider the repeated integrals of $|g|$, using whichever order we want (as long as the measures meet the conditions of Tonelli).

1.4 Show that if μ and γ are both σ-finite, then $\mu \otimes \gamma$ must also be σ-finite.

1.5 The *product-measurability* assumption for f in Tonelli and Fubini is important. Devise an example in which the two orders of iterated integrals are both well-defined and finite but disagree with each other.

It's usually inconvenient to directly check a function for product measurability, but there are various simple conditions that automatically imply it — recall Section REF.

1.6 Let $\{\phi_\omega : \omega \in \Omega\} \subseteq C(\mathbb{M})$ be a family of continuous real-valued functions defined on a compact metric space $\mathbb{M}$. Assume μ is a measure on (Ω, Σ) for which the mapping $f : \omega \mapsto \phi_\omega$ has $\mu \|f\|_{C(\mathbb{M})} < \infty$ and that for every $x \in \mathbb{M}$, $\omega \mapsto \phi_\omega(x)$ is Σ-measurable. Explain why the point-wise integral

$$\int_\Omega \phi_\omega(x) d\mu(\omega)$$

is the Bochner integral of f with respect to μ.

Tonelli and Fubini are often invoked to justify interchanges in the order of integration, a marvelously useful trick.

1.7 Let X be a non-negative random variable. Prove that

$$\mathbb{E}_L X = \int_0^\infty \mathbb{P}\{X > t\} dt.$$

2 Mixtures

Of special importance is the case in which the measure being "multiplied" with a σ-finite kernel Φ is a probability measure P. In that context, we will let $\bar{\Phi}_P$ denote the marginal measure of $P \otimes \Phi$ for $(\mathbb{Y}, \mathbb{B})$; we call $\bar{\Phi}_P$ the *P-mixture over* Φ and call P the *mixing measure.*

1.8 Suppose that all of the measures in Φ assign the same measure to the full space $\mathbb{Y}$. Show that $\bar{\Phi}_P$ also assigns that same measure to $\mathbb{Y}$.

Given a family $\Phi := \{\Phi_x : x \in \mathbb{X}\}$ of measures on $(\mathbb{Y}, \mathbb{B})$, one may not have any particular σ-algebra "in mind" for $\mathbb{X}$. We will use the notation $\sigma(\Phi)$ to denote the coarsest σ-algebra for which Φ is a kernel from $(\mathbb{X}, \sigma(\Phi))$ to $(\mathbb{Y}, \mathbb{B})$. This "coarsest" σ-algebra is well-defined because the power set works and the intersection of σ-algebras is a σ-algebra. Any probability measure P on $\mathbb{X}$ whose domain σ-algebra is at least as fine as $\sigma(\Phi)$ can be used to create a well-defined mixture over Φ.

1.9 Let $(\mathbb{Y}, \mathbb{B}, \mu)$ be a measure space and $\{\phi_x : x \in \mathbb{X}\}$ be a family of functions in $\mathcal{L}^1(\mu)$. Assume $(x, y) \mapsto \phi_x(y)$ is $(\mathbb{A} \otimes \mathbb{B})$-measurable for *some* $\mathbb{A}$. Show that if P is a probability measure on $\mathbb{X}$ with domain at least as fine as $\mathbb{A}$, and $\mathbb{E}_{X \sim P} \|\phi_X\|_1 < \infty$, then $(x, y) \mapsto \phi_x(y)$ is $(P \otimes \mu)$-integrable.

1.10 Suppose that a family Φ of finite measures on $(\mathbb{Y}, \mathbb{B})$ has densities $\{\phi_x : x \in \mathbb{X}\}$ with respect to a measure μ. if there exists a σ-finite dominating measure μ, then densities with respect to μ exist by the Radon-Nikodym Theorem. REF - except i don't think they necessarily have the product-measurability I need! Assume $(x, y) \mapsto \phi_x(y)$ is $(\mathbb{A} \otimes \mathbb{B})$-measurable for *some* $\mathbb{A}$. Show that if P is a probability measure on $\mathbb{X}$ with domain at least as fine as $\mathbb{A}$, then $\bar{\Phi}_P$ has density

$$\bar{\phi}_P(y) = \int_{\mathbb{X}} \phi_x(y) dP(x)$$

with respect to μ.

Additional discussion of integral transforms

3 Disintegration

Given a measure space $(\Omega, \mathbb{A}, \mu)$ and a measurable subset B, we define the *restricted measure* μ_B by setting $\mu_B A := \mu(A \cap B)$ for every $A \in \mathbb{A}$.

1.11 Let $\{B_i\}$ be a countable collection of measurable sets that partitions Ω. Show that for any $f \in \mathbb{M}^+$, $\mu f = \sum_i \mu_{B_i} f$.

The idea described in Exercise 1.11 can be thought of as introducing a product space for which an iterated integral is the same as the original μ-integral on $(\Omega, \mathbb{A})$. This product space comprises the index set of the $\{B_i\}$ (with its power set as σ-algebra) times the original $(\Omega, \mathbb{A})$. The original measure gets "arranged" onto the product space in such that it's restriction to B_i is exactly the measure on "slice" i.

This process of splitting up a measure into components is a sort of "inverse" operation to the creation of a mixture, with the additional desire that each measure in the family should "live on" its own piece of some partitioning of the original space $\mathbb{Y}$. Based on this insight, let's look to generalize the construction from Exercise 1.11.

Continue coverage of disintegration and then conditional distributions and conditional expectations.