Integrals

You'll recall that the *Reimann integral* is defined as a limiting sum of areas of *vertical* slices under a real-valued function. Some decades after Bernhard Reimann formulated his integral, *Henri Lebesgue* turned that idea on its side: he defined an integral as a limiting sum of areas of *horizontal* slices. Lebesgue's invention was later modified by Salomon Bochner into a notion of integral that can be applied to functions that map from a measure space to a Banach space. *Israel Gelfand* and *B.J. Pettis* abstracted the concept even further. In this section, we describe the most prominent measure-theoretic notions of *integral* and cover some of their properties.

1 Defining integrals

In our exploration of measure spaces, we saw how a premeasure μ defined on a semi-ring of subsets can be naturally extended to the completion of the sigma field generated by that semi-ring. An equivalent formulation instead considers μ to be a functional (called an *integral*) defined on indicator functions of those subsets: $\mu\,\mathbb{I}_A \equiv \mu\,A$. This idea motivates us to *continue extending the domain of μ to more functions if it naturally assigns them a value.*

1.1 The discrete integral

Intuitively, the idea of the integral is to calculate a "total" value in common applications. For instance, consider making a salad by mixing m different ingredients, indexed by $\{1,\ldots,m\}$. Let μ be a measure (with the power set $2^{\{1,\ldots,m\}}$ as its domain) that maps $\{i\}$ to the number of servings of ingredient i that the salad contains. Define a function f by letting $f(i)$ equal the number of calories per serving of ingredient i. To find the total number of calories in the salad, we simply need to sum over the ingredients, "weighting" the calories per serving by the number of servings:

$$\sum_{i=1}^{m} [\mu\,\{i\}] f(i)$$

In general, a more convenient formulation is to instead sum over the different possible values in the range of f. This change in perspective hints at the difference between Reimann integrals and Lebesgue integrals. For instance, in some situations the domain of f could have infinite cardinality while its range has finite cardinality; then we still have a finite sum over the values in the range $\{v_1,\ldots,v_n\}$.

$$\mu\,f := \sum_{i=1}^{n} [\mu\,\{f^{-1}(v_i)\}] v_i.$$

As long as we know what measure to assign to the pre-image of each x_i and those measure values are finite, then we can calculate this total. By convention, if zero (or, more generally, the zero vector) is in the range of f, we omit it from the sum, even if its pre-image has infinite measure. We *take the above as the definition of the integral* with respect to μ of any function f that maps from Ω to finitely many values in a real or complex vector space $\mathbb{V}$, assuming each $\{f^{-1}(v_i)\}$ is in $\bar{\Sigma}(\mu)$ and has finite measure for non-zero v_i. If any of those pre-images are not in $\bar{\Sigma}$, then the integral is undefined. When $\mathbb{V}$ is $\mathbb{R}$, we sometimes allow for more sets of infinite measure, as you will see in the Lebesgue integral construction. Otherwise, we leave the integral undefined if the pre-image of any non-zero v_i has infinite measure.

If $\mathbb{V}$ also has a topology, then it is natural to extend this idea to functions that take a countable infinity of values by defining

$$\mu\,f := \sum_{i} [\mu\,\{f^{-1}(v_i)\}] v_i$$

if the pre-images are all in $\bar{\Sigma}$ and the infinite sum is unconditionally convergent.

1.1 Explain why the integral of a discretely-valued function from Ω to $\mathbb{V}$ commutes with all continuous linear functionals. Explain also why *unconditional* convergence ensures that there is only one vector v_f for which $l(v_f) = \mu\,l(f)$ for all $l \in \mathbb{V}'$.

B. J. Pettis extended the notion of an integral by taking *commutation with continuous linear functionals* to be the *defining* property. But before we can make sense of his formulation, we need to learn about Lebesgue's integral for $\bar{\mathbb{R}}$-valued functions.

1.2 The Lebesgue integral for $\bar{\mathbb{R}}$-valued functions

The Lebesgue integral on $\bar{\mathbb{R}}$-valued functions plays a central role in the construction of the Bochner integral and in the definition of the Pettis integral for vector-valued functions. Those two concepts turn out to be *almost but not quite* generalizations of the Lebesgue integral. We can already see one difference: $\bar{\mathbb{R}}$ is *not a vector space.* Why do we care about including $\pm\infty$? One reason is that a sequence of real-valued functions may have a point-wise limit that is infinite in some places; it is convenient that the Lebesgue integral can handle such point-wise limits. The same goes for point-wise infimum and point-wise supremum.

Let (Ω,Σ,μ) be a measure space. We start our definition of the Lebesgue integral with the most "obvious" cases: the *Lebesgue integral* of any measurable set's indicator function equals the measure of that set, even if it is infinity. Next, we take a baby step by extending the definition to *non-negative simple functions.* Suppose $s \in \mathbb{S}^+(\Omega,\Sigma)$ has a representation

$$s := \sum_{i=1}^{n} \mathbb{I}_{A_i} x_i$$

where each $A_i \in \Sigma$ and each $x_i \in \mathbb{R}^+$. The *Lebesgue integral* of s with respect to μ is defined to be

$$\mu_L s := \sum_{i=1}^{n} (\mu\,A_i) x_i.$$

1.2 Let's establish that the Lebesgue integral is well-defined for the non-negative simple functions: explain why the value of $\mu_L s$ does not depend on the choice of representation for s.

Now we propose a new target: $M^+(\Omega,\Sigma)$ (hereafter denoted M^+). We define the Lebesgue integral of $f \in M^+$ to be

$$\mu_L f := \sup_{s \le f} \mu_L s \tag{1}$$

where the supremum is taken over all the functions in $\mathbb{S}^+$ that are everywhere upper bounded by f. Again, $\mu_L f$ may be infinite.

Why did we limit the definition to the strongly measurable functions? M^+ comprises exactly the non-negative functions that can be approximated point-wise by a sequence of simple functions. Thus if f maps from Ω to $\bar{\mathbb{R}}^+$ but is not in M^+, then there must be gaps between f and $\sup_{s\le f} s$. However, when $f \in M^+$, we have $f = \sup_{s \le f} s$, as you will verify in Exercise 1.3.

1.3 Devise a point-wise non-decreasing sequence of simple functions that approaches $f \in M^+$ at every $\omega \in \Omega$.

1.4 If $a, b \in \mathbb{R}^+$ and $f, g \in M^+$, show that $\mu_L(af + bg) = a\mu_L f + b\mu_L g$.

From Exercise 1.3, every $f \in M^+$ can be approximated from below by simple functions. It is comforting to learn that *every* sequence of simple functions converging upward to f has the same limit of integrals, and that limit is $\mu_L f$.

Theorem 1.1 (Monotone Convergence Theorem, see Pollard, 2002, Section 2.4)**.** *Suppose (f_n) is a non-decreasing sequence of functions in M^+. Then $\lim f_n \in M^+$ and $\mu_L f_n \uparrow \mu_L \lim f_n$.*

Interestingly, it turns out that every functional on M^+ that behaves like a Lebesgue integral in a few basic ways *is a Lebesgue integral.*

Proposition 1.2 (see Pollard, 2002, Theorems 2.12 and 2.13)**.** *Suppose γ is a mapping from $M^+(\Omega,\Sigma)$ to $\bar{\mathbb{R}}^+$ such that*

(i) γ maps the zero function to 0

(ii) for $a, b \in \mathbb{R}^+$, $\gamma\,(af + bg) = a\gamma\,f + b\gamma\,g$

(iii) $f \le g$ implies $\gamma\,f \le \gamma\,g$

(iv) γ has the Monotone Convergence property. In other words, γ could be substituted in place of μ_L in the Monotone Convergence Theorem, and the statement would be true.

Then γ is the Lebesgue integral corresponding to the measure on Σ defined by $A \mapsto \gamma\,\mathbb{I}_A$. Conversely, every Lebesgue integral corresponding to a measure on (Ω,Σ) has the four properties listed.

Our next target is $M(\Omega,\Sigma)$ (hereafter denoted M). Any real-valued function f can be expressed as the difference $f_+ - f_-$ where $f_+ := f \vee 0$ and $f_- := (-f) \vee 0$. We define $\mu_L f$ to be $\mu_L f_+ - \mu_L f_-$ unless both of these quantities are infinity. In that case, the Lebesgue integral of f is undefined.

1.5 For $f \in M$, how do we know that f_+ and f_- are in M^+?

1.6 Suppose $f \in M$ has a well-defined Lebesgue integral. Show that for any $g \in M$ with $f \le g$ (everywhere), $\mu_L f \le \mu_L g$. In other words, you're showing that the Lebesgue integral is an increasing functional on the set of M for which it is well-defined. The partial ordering is defined by point-wise comparisons of the functions.

1.7 Suppose $f, g \in M$ have well-defined Lebesgue integrals. Prove that $\mu_L(f+g) = \mu_L f + \mu_L g$ iff $\mu_L f$ and $\mu_L g$ aren't infinite with opposite sign.

1.8 Prove the Borel-Cantelli Lemma (BCL): If $f \in M$ is integrable, then the set where $f(\omega) \in \{-\infty, \infty\}$ must be negligible.

1.9 Prove Fatou's Lemma: If (f_n) is a sequence in M^+, then

$$\mu_L \liminf f_n \le \liminf \mu_L f_n$$

1.10 Let $f, g \in M$ be in the same μ-equivalence class. Show that they must have the same Lebesgue integral.

Finally, there is one last step in our extension. As described in REF, it is perfectly natural to extend the measure μ to sets in $\bar{\Sigma}(\mu)$. Thus, we should not hesitate to apply the Lebesgue integral to functions in $M(\Omega,\bar{\Sigma})$.

1.11 True or false: $g \in M(\Omega,\bar{\Sigma})$ iff there exists an $f \in M(\Omega,\Sigma)$ that disagrees with g only on a μ-subnegligible set.

Defining $\mathrm{L}^p_{\mathbb{X}}$-norms and spaces

1.12 Given a real Banach space $\mathbb{X}$, explain why the *integrated norm* $\|f\|_{L^1_{\mathbb{X}}} := \mu_L\|f\|$ is well-defined for any $f \in M_{\mathbb{X}}(\Omega,\Sigma)$. What about a generalization of this, $\|f\|_{L^p_{\mathbb{X}}} := [\mu_L\|f\|^p]^{1/p}$ with $p \in [1,\infty]$? The $p = \infty$ case needs clarification: $\|f\|_{L^\infty_{\mathbb{X}}}$ is defined to be the essential supremum of f. This is consistent with the limiting behavior as $p \to \infty$. Show that $\|\cdot\|_{L^p_{\mathbb{X}}}$ is absolutely homogeneous.

Theorem 1.3 (Minkowski's inequality, CITE SOURCE)**.** *For any $p \in [1,\infty]$ and $f, g \in M_{\mathbb{X}}(\Omega,\Sigma)$,*

$$\|f+g\|_{L^p_{\mathbb{X}}} \le \|f\|_{L^p_{\mathbb{X}}} + \|g\|_{L^p_{\mathbb{X}}}$$

As defined in Exercise 1.12, the functional $\|\cdot\|_{L^p_{\mathbb{X}}}$ on $M_{\mathbb{X}}$ is called the $L^p_{\mathbb{X}}(\mu)$-norm (or the $L^p_{\mathbb{X}}$-norm if the measure is clear from context or the $L^p_{\mathbb{X}}(\Omega,\Sigma,\mu)$-norm if the measurable space is not clear from context). Notice that the only role of the measure in $L^\infty(\mu)$ is in determining which sets are negligible. Exercise 1.12 and Minkowski's inequality together show us that $\|\cdot\|_{L^p_{\mathbb{X}}}$ is a semi-norm on $M_{\mathbb{X}}$. It is of course a true norm on the Kolmogorov quotient, that is, the equivalence classes of $M_{\mathbb{X}}$ in which equivalence of f and g means $[\mu_L\|f-g\|^p]^{1/p} = 0$. The same name $L^1_{\mathbb{X}}$*-norm* and notation is used for the semi-norm on functions and the norm on equivalence classes of functions. Usually the distinction between functions and their equivalence classes is unimportant anyway.

1.13 Explain why these equivalence classes are exactly the same as the μ-equivalence classes defined in Section REF.

The subset of $M_{\mathbb{X}}$ with finite $L^p_{\mathbb{X}}(\mu)$-norm is a semi-normed space called $\mathbb{L}^p_{\mathbb{X}}(\Omega,\Sigma,\mu)$; the normed space of μ-equivalence classes is denoted $L^p_{\mathbb{X}}(\mu)$. If the relevant measure space is not clear from context, one should include it in the notation: $\mathbb{L}^p_{\mathbb{X}}(\Omega,\Sigma,\mu)$ and $L^p_{\mathbb{X}}(\Omega,\Sigma,\mu)$. On the other hand, when the measure space is clear, one can write $\mathbb{L}^p_{\mathbb{X}}$ and $L^p_{\mathbb{X}}$. When the codomain is $\mathbb{R}$, we omit the subscript, writing simply $\mathbb{L}^p(\mu)$ and $L^p(\mu)$.

Theorem 1.4 (Fischer-Riesz Theorem, see Neerven, 2010, Section 1.3.2)**.** *Given a measure space (Ω,Σ,μ), a Banach space $\mathbb{X}$, and any $p \in [1,\infty]$, the normed space $L^p_{\mathbb{X}}(\Omega,\Sigma,\mu)$ is complete.*

Even though it is only a semi-normed space, we will still speak of *convergence* in $\mathbb{L}^p_{\mathbb{X}}$ spaces. Based on the topological definition of limits, there is nothing improper about this. The notation $f_n \xrightarrow{L^p_{\mathbb{X}}} f$ means $\|f - f_n\|_{L^p_{\mathbb{X}}} \to 0$. If a sequence converges to a function in this semi-normed space, it also converges to all of the other functions in that same μ-equivalence class.

Theorem 1.5. *Let (f_n) be a sequence of functions in $\mathbb{L}^p_{\mathbb{X}}$. If $f_n \xrightarrow{L^p_{\mathbb{X}}} f$, then there exists a subsequence of (f_n) that converges uniformly to f almost everywhere.*

For $p = \infty$, the relationship is much stronger. $L^\infty_{\mathbb{X}}$ convergence is equivalent to uniform convergence almost everywhere.

1.14 Use the completeness of $L^p_{\mathbb{X}}$-spaces to prove Theorem 1.5.

1.15 Devise an example in which $f_n \xrightarrow{L^1_{\mathbb{X}}} f$ but (f_n) does not converge almost everywhere to f.

1.16 Devise an example in which (f_n) converges point-wise to f but it is not the case that $f_n \xrightarrow{L^1_{\mathbb{X}}} f$.

1.17 A sequence of functions (f_n) converges in measure to f (written $f_n \xrightarrow{\mu} f$) if for every $\epsilon > 0$, $\mu\,\{\|f_n - f\| > \epsilon\} \to 0$. Explain why almost-everywhere convergence implies convergence in measure. Devise an example of convergence in measure without almost-everywhere convergence.

We see from Exercises 1.15 and 1.16 that the relationship between $L^1_{\mathbb{X}}$ convergence and point-wise convergence is a bit complicated. Neither type of convergence implies the other. But we also know from Theorem 1.5 that $L^1_{\mathbb{X}}$ convergence *does imply* the existence of a subsequence that converges almost everywhere to the $L^1_{\mathbb{X}}$-limiting function. Going the other direction, the Dominated Convergence Theorem (DCT) provides a sufficient condition for point-wise convergence to imply $L^1_{\mathbb{X}}$ convergence. A function g is said to *dominate* another function if g upper bounds that function on its entire domain. To say that *g dominates a set of functions* means that it dominates every one of them individually.

Theorem 1.6 (Dominated Convergence Theorem)**.** *Suppose a sequence (f_n) in $\mathbb{L}^1_{\mathbb{X}}$ converges point-wise to f. If there exists a function g that dominates the sequence $(\|f_n\|)$ and has $\mu_L g < \infty$, then $f_n \xrightarrow{L^1_{\mathbb{X}}} f$.*

1.18 Use Fatou's Lemma to prove the DCT for $\mathbb{L}^1(\mu)$. Use that result to prove the general DCT.

1.19 Show that the simple functions that belong to $\mathbb{L}^p_{\mathbb{X}}(\mu)$ comprise a dense subspace of $\mathbb{L}^p_{\mathbb{X}}(\mu)$.

We will have much more to say about $\mathbb{L}^p_{\mathbb{X}}$ spaces in Chapter REF.

1.3 The Bochner integral for $\mathbb{X}$-valued functions

The Lebesgue integral enabled us to extend a measure to a class of $\bar{\mathbb{R}}$-valued functions. Next, we construct the *Bochner integral* (or *strong integral*) which applies the same thinking to functions mapping to any real or complex Banach space. Throughout the remainder of this section, assume $\mathbb{X}$ is a *real* Banach space. If a function of interest maps from Ω to a normed space that isn't necessarily complete, then you can always consider the *completion of that normed space* as the function's codomain. Thus the Bochner integral theory we develop here can still be applied accordingly.

Again, we begin with the simple functions. Let $s \in \mathbb{S}_{\mathbb{X}}(\Omega,\Sigma)$ have representation

$$s(\omega) = \sum_{i=1}^{n} \mathbb{I}_{A_i}(\omega) x_i, \tag{2}$$

If each of these A_i has finite measure, then the *Bochner integral* of s is defined to be

$$\mu\,s := \sum_{i=1}^{n} (\mu\,A_i) x_i;$$

otherwise the Bochner integral $\mu\,s$ is undefined. We denote the subset of $\mathbb{S}_{\mathbb{X}}$ with Bochner integrals by $\mathbb{S}^1_{\mathbb{X}}(\mu)$.

1.20 Explain why no simple function has a *unique* representation of the form (2). Show that $\mu\,s$ does not depend on the choice of representation.

We now extend this integral to the $L^1_{\mathbb{X}}$-closure of $\mathbb{S}^1_{\mathbb{X}}$. Let f be a function from Ω to $\mathbb{X}$. If there exists a sequence (s_n) in $\mathbb{S}^1_{\mathbb{X}}$ with $s_n \xrightarrow{L^1_{\mathbb{X}}} f$, then the Bochner integral of f is $\mu\,f := \lim_{n\to\infty} \mu\,s_n$. Otherwise the Bochner integral of f is undefined. Any function with a Bochner integral is called Bochner integrable.

1.21 Let's make sure that our definition of $\mu\,f$ makes sense. Show that the the convergent sequence (s_n) of simple functions must have a limit of Bochner integrals. Show also that if there exists any other satisfactory sequence in $\mathbb{S}^1_{\mathbb{X}}$, it must have the same limit of Bochner integrals.

1.22 Explain why every Bochner integrable function must be in $M_{\mathbb{X}}$.

Importantly, the set of Bochner integrable functions is exactly $\mathbb{L}^1_{\mathbb{X}}$.

Theorem 1.7. *f is Bochner integrable iff $f \in \mathbb{L}^1_{\mathbb{X}}$.*

The properties described for $\mathbb{S}^1_{\mathbb{X}}$ sequences in Exercise 1.21 actually hold more generally.

Theorem 1.8. *Let (f_n) be a sequence of functions in $\mathbb{L}^1_{\mathbb{X}}$. If $f_n \xrightarrow{L^1_{\mathbb{X}}} f$, then f is Bochner integrable and $\mu\,f_n \to \mu\,f$.*

Theorems 1.7 and 1.8 indicate that the theory of Bochner integration is intimately connected to $L^1_{\mathbb{X}}$ spaces.

1.23 Prove Theorem 1.7.

1.24 Prove Theorem 1.8.

It is convenient that for Bochner integrable real-valued functions, our theory of Bochner integration subsumes the theory of Lebesgue integration.

1.25 Explain why $f : \Omega \to \mathbb{R}$ is Bochner integrable iff its Lebesgue integral is finite. Show that the two integrals coincide ($\mu\,f = \mu_L f$) in that case.

1.26 Show that the Bochner integral "commutes" with continuous linear functionals. That is, if f is a Bochner integrable $\mathbb{X}$-valued function, then for any $l \in \mathbb{X}'$

$$\mu\,l(f) = l(\mu\,f)$$

1.4 The Pettis integral for $\mathbb{V}$-valued functions

Now, we return to the even more general setting where $\mathbb{V}$ is a real topological vector space, and think about functions that aren't necessarily discretely-valued. Let $f : \Omega \to \mathbb{V}$ be a weakly measurable function. If there exists a unique $v_f \in \mathbb{V}$ such that

$$l(v_f) = \mu\,l(f) \qquad \text{for every } l \in \mathbb{V}', \tag{3}$$

then we call v_f the Pettis integral (or *weak integral*) of f. In plan English, the Pettis integral of f is a unique vector that can be used "in place of" $f(\omega)$ without affecting the values of the Lebesgue integrals of any continuous linear functionals composed with f. In fact, whenever we say *integral*, we mean *Pettis integral.* When f has a (Pettis) integral, we will call it an integrable function. What we're calling the *Pettis integral* is sometimes called the *Gelfand-Pettis integral.* Notice that if f is integrable, then it must of course be weakly measurable.

1.27 Suppose the dual space $\mathbb{V}'$ separates points. Show that an $v_f \in \mathbb{V}$ satisfying (3) is unique if it exists. Recall that the dual space of any LCHS separates points - see Exercise REF

We've seen in Exercise 1.26 that Bochner integrals commute with continuous linear functionals, so we can conclude that every Bochner integral is a Pettis integral. The dual space of a normed space separates points, so any Bochner integral is indeed a *unique* vector satisfying (3). Likewise, Exercise 1.1 shows that discrete integrals on topological vector spaces are Pettis integrals as well. When we considered integrals of measurable functions taking finitely many values, we did not assume any topology on the vector space; in those cases, the discrete integral is a Pettis integral for every possible topology on the space since all linear operators commute with a finite sum by definition.

Thus, the Pettis integral definition can be seen as another logical step in extending a measure to more functions than the Bochner construction reached; we will continue to use the same notation (e.g. μf) for Pettis integrals. The term *integral* can be used to refer to either the operator on the space of functions or to that operator's output.

1.28 Show that a real-valued function is integrable iff it is Bochner integrable. Taking this fact in conjunction with Exercise 1.25, we see that the Lebesgue, Bochner, and Pettis integrals all coincide for any function with a finite Lebesgue integral.

There are cases where a function is not Bochner integrable but it is (Pettis) integrable. One obvious example of the limitations of Bochner integral theory: it is clear from our construction and from the Pettis Measurability Theorem that every Bochner integrable function has to be essentially separably-valued. To see this, let's compare the Bochner and Pettis integrals in the context of a countably-infinitely-valued functions.

1.29 Let $f(\omega) := \sum_i \mathbb{I}_{A_i}(\omega) x_i$ be a series with each $A_i \in \Sigma$ and each x_i in the Banach space $\mathbb{X}$. Based on Exercise 1.1, we see that f is Pettis integrable iff it is unconditionally convergent and also that the Pettis integral is $\sum_i (\mu\,A_i) x_i$ if it exists. Show that f is Bochner integrable iff the sum $\sum_i (\mu\,A_i) x_i$ is absolutely convergent.

By the Dvoretsky-Rogers Theorem (Theorem **??**), we can conclude that every discrete integral in a finite-dimensional Banach space is a Bochner integral. The theorem also tells us that in every infinite-dimensional Banach space, there exists a sequence that is unconditionally convergent but not absolutely convergent. Based on Exercise 1.29, we realize that for any such sequence, one could construct a measure space and function for which the Pettis integral exists while the Bochner integral doesn't.

Let's take a moment to think about the role of the codomain's topology in the Pettis integral definition. The *linearity* of a functional is purely algebraic, but the *continuity* has everything to do with topology. Suppose v_f is the Pettis integral when using topology τ_1. Will it remain the Pettis integral for a coarser topology $\tau_0 \subseteq \tau_1$? It is certainly the only candidate, as it still commutes with all of the (now possibly fewer) continuous linear functionals. The only thing that can go wrong is that it may no longer be a *unique* satisfier of (3), though that is guaranteed by, for instance, an LCHS structure governing τ_0. What about if we consider a finer topology $\tau_2 \supseteq \tau_1$? Again, v_f is the only candidate, but we need to figure out whether or not there still exists a Pettis integral, as there may now be more continuous linear functionals.

In the next few exercises, we will identify contexts in which we can be more explicit about what form a Pettis integral takes when it exists.

1.30 Suppose f maps from (Ω,Σ,μ) to a TVS of functions from a set $\mathbb{U}$ to a real TVS $\mathbb{V}$; we will write $f(\omega)$ as ϕ_ω in order to more intuitively think of the range of f as a family of functions $\{\phi_\omega : \omega \in \Omega\}$ in the function space. Assume the topology we're using on this space of functions is fine enough that the point-evaluation functionals l_u are all continuous. Explain why if f is integrable, its Pettis integral must be equal to the point-wise integral:

$$[\mu\,f](u) = \int \phi_\omega(u) d\mu(\omega).$$

1.31 Let $\mathbb{X}$ be a Banach space with a Schauder basis $B := (b_1, b_2, \ldots)$. Let f be an $\mathbb{X}$-valued function, and let $(f_1, f_2, \ldots)$ be its coefficients with respect to B. That is, for any $\omega \in \Omega$, the $f_i(\omega)$ are real numbers defined by

$$f(\omega) = \sum_i f_i(\omega) b_i.$$

Explain why if f is integrable, then

$$\mu\,f = \sum_i (\mu\,f_i) b_i.$$

1.32 Is there a result analogous to that of Exercise 1.31 for Hamel bases?

Indefinite integrals

If fA is integrable for every $A \in \mathcal{A}$ then we will call f indefinitely integrable. What we call *indefinite integrability* is taken to be the definition of "integrability" in the context of Pettis integral by many authors. then the mapping from $\mathcal{A}$ to $\mathcal{V}$ defined by

$$A \mapsto \mu\,fA \tag{4}$$

is called the indefinite integral of f with respect to μ. We may also denote $\mu\,fA$ by $\mu_A\,f$ to highlight the idea that it is just the ordinary integral after restricting μ and f to $A \subseteq \Omega$. f is called a density with respect to μ for the vector measure satisfying (4). Another term for *density* is *Radon-Nikodym derivative.*

1.33 Show that the mapping (4) is indeed a vector measure and that it is absolutely continuous with respect to μ.

1.34 Show that every Bochner integrable function is indefinitely Bochner integrable (i.e. for every $A \in \mathcal{A}$, fA is Bochner integrable).

1.35 Show that two densities with respect to μ define the same measures iff they are μ-equivalent.

1.36 When f is Bochner integrable, explain why the vector measure defined by (4) must have finite total variation.

2 Properties of integrals

We've seen that the Pettis integral generalizes the notions of discrete integrals and Bochner integrals. For the sake of efficiency, we will first cover some properties that hold for all Pettis integrals before moving on to more specific properties of Bochner integrals.

2.1 Properties of Pettis integrals

The following exercises highlight some of the key properties of Pettis integrals.

1.37 Show that if f and g are μ-equivalent, then their Pettis integrals are equal. In other words, if one of the functions has its Pettis integral defined, the other function's Pettis integral is also defined and is equal to the first. Consequently, if one is undefined, the other must also be undefined.

1.38 Show that the set of Pettis integrable functions from (Ω,Σ,μ) to a real TVS $\mathbb{V}$ is a subspace and that the Pettis integral is a linear operator on this space.

1.39 Suppose f is a $\mathbb{V}$-valued Pettis integrable function, and T is a continuous linear operator from $\mathbb{V}$ to $\mathbb{W}$. Show that $T \circ f$ is weakly measurable and that $\mu\,(T \circ f) = T\mu\,f$.

The Pettis integral operator commutes not only with continuous linear functionals but *with any continuous linear operator* that is acting on a Pettis integrable function. Furthermore, the Pettis integral operator is itself a linear operator.

1.40 Let μ be Lebesgue measure on $\mathbb{R}$, and let $k \in C(I \times J)$ where I and J are real intervals. For any $g \in C(J)$, define Kg by

$$(Kg)(x) = \int_J k(x,y) g(y) d\mu(y)$$

for every $x \in I$. The operator K is called an integral transform, and k is called its kernel. Show that K is a continuous linear operator from $C(J)$ to $C(I)$. Provide an interpretation in terms of a Pettis integral.

1.41 Let $\mathcal{I}$ be an interval of the real line. Let $\omega \mapsto \phi_\omega$ be a mapping from (Ω,Σ) to the Banach space of continuously differentiable real-valued functions with domain I and with finite $C^1(I)$-norm. Suppose the Pettis integral of $\omega \mapsto \phi_\omega$ with respect to μ exists. Explain why for all $x \in \mathcal{I}$,

$$\frac{d}{dx}\int_{\mathbb{X}} \phi_\omega(x) d\mu(\omega) = \int_{\mathbb{X}} \frac{\partial}{\partial x}\phi_\omega(x) d\mu(\omega)$$

and why the resulting function on I is continuous and bounded.

1.42* Another technique to justify derivative-integral interchanges is by using the Dominated Convergence Theorem to pass a limit through an integral. In this exercise, use the DCT to prove a measure theoretic version of the Leibniz integral rule: Let $\{\phi_\omega : \omega \in \Omega\}$ be a family of differentiable real-valued functions defined on an open interval $\mathcal{I} \subseteq \mathbb{R}$. Suppose μ is a measure on Ω for which $\omega \mapsto \phi_\omega(x)$ is integrable for every $x \in \mathcal{I}$. Suppose also that there exists an integrable real-valued function $g : \Omega \to \mathbb{R}^+$ such that $|\frac{\partial}{\partial x}\phi_\omega(x)| \le g(\omega)$ for all $\omega \in \Omega$. Show that for all $x \in \mathcal{I}$,

$$\frac{d}{dx}\int_{\mathbb{X}} \phi_\omega(x) d\mu(\omega) = \int_{\mathbb{X}} \frac{\partial}{\partial x}\phi_\omega(x) d\mu(\omega).$$

1.43* Using a supremely clever trick, evaluate $\int_0^\infty x^2 e^{-x}\,dx$.

1.44 Let f be a strongly measurable function from (Ω,Σ) to a Banach space $\mathbb{X}$. Prove that if f is integrable with respect to μ, then $\|\mu\,f\| \le \mathcal{M}_L\|f\|$.

2.2 Properties of Bochner integrals

Every Bochner integral is also a Pettis integral, so it has the properties we have already covered. Here we point out a few additional properties that Bochner integrals have.

A variation on the result of Exercise 1.39 provides a different condition for linear operators to commute with Bochner integrals. A linear operator T from one TVS to another is called a closed linear operator it has the property that for any convergent sequence v_n in its domain, the sequence it maps to has $\lim Tv_n = T\lim v_n$ *if that sequence converges.* This is weaker than continuity because the sequence of range values is allowed to diverge.

Theorem 2.1 (Hille's Theorem, see Neerven, 2010, Theorem 1.19)**.** *Let f map from (Ω,Σ,μ) to a Banach space $\mathbb{X}$, and let T be a closed linear operator from $S \subseteq \mathbb{X}$ to a Banach space $\mathbb{Y}$. Suppose the range of f is a subset of S. If f and $T \circ f$ are both Bochner integrable, then $\mu\,f \in S$ and*

$$\mu\,(T \circ f) = T\mu\,f$$

We've seen that every Bochner integrable function is the density for some vector measure with finite total variation that is dominated by μ (Exercise REF). We say that a Banach space $\mathbb{X}$ has the *Radon-Nikodym property with respect to μ* if a conditional converse to this holds: every $\gamma \in \mathbb{M}_{\mathbb{X}}(\Omega,\mathcal{A})$ dominated by μ has a Bochner integrable density with respect to μ. We say that $\mathbb{X}$ has the Radon-Nikodym property if it has the Radon-Nikodym property with respect to every finite measure.

Theorem 2.2. *Any of the following conditions are sufficient for a Banach space to have the Radon-Nikodym property:*

- *it is reflexive,*
- *its dual space is separable,*
- *it is the l^1-space.*

Still need to write this section...

2.3 Properties of Lebesgue integrals

When a real-valued function is integrable, it is Bochner integrable and thus has all of the properties we have already covered. Here, we will explore some additional properties that hold specifically for real-valued (as well as $\bar{\mathbb{R}}$-valued) functions.

- use term indefinite Lebesgue integral as well - how does it work in that case? - only need to worry about case when $\mu_L f$ is $\pm\infty$ - the other cases are already taken care of by the more general result, right? - from Fremlin Prop 232D - show that it is a signed measure and that it is truly continuous with respect to μ

Theorem 2.2 provided conditions that guarantee the existence of densities for vector measures with finite total variation. For signed measures, we have a more convenient result that doesn't require finite total variation.

Theorem 2.3 (Radon-Nikodym Theorem, Fremlin, 2000, Theorem 232E)**.** *Let $(\Omega,\mathcal{A},\gamma)$ be a measure space. If μ is a signed measure that is truly continuous with respect to γ, then there exists an integrable function f such that for every $A \in \mathcal{A}$,*

$$\mu\,A = \gamma_A\,f$$

By Exercise ??, if the dominating measure is σ-finite, then absolute continuity is sufficient for the existence of a density. In fact, that is the usual statement of the Radon-Nikodym Theorem.

Still need to write this section...

3 Existence of integrals

We've seen a number of properties that hold "when the integral exists." Here we collect additional observations that can be useful when trying to guarantee existence.

1.45 Are there converse statements to Exercise 1.31 and 1.32 that can guarantee integrability of f when all the coefficient functions f_i are integrable?

1.46 We say that (f_n) *converges weakly in measure* to f if every continuous linear functional l has $l \circ f_n$ converging in measure to $l \circ f$. Suppose (f_n) is a sequence of integrable functions converging weakly in measure to f and that $\lim \mu\,f_n$ exists. Show that $\mu\,f = \lim \mu\,f_n$.

One might hope that the integrability of all the continuous linear functionals of a function would imply its integrability. At least in the context of probability spaces a slightly stronger condition is sufficient. You will see in Section ?? why a finite $L^p(\mathbb{P})$-norm with $p > 1$ is a stronger requirement than a finite $L^1(\mathbb{P})$-norm.

Theorem 3.1 (see Neerven, 2010, Theorem 1.20)**.** *Let X be a strongly measurable function from a probability space $(\Omega,\Sigma,\mathbb{P})$ to a Banach space $\mathbb{X}$. If there exists a fixed $p \in (1,\infty]$ for which $(l \circ X) \in L^p(\mathbb{P})$ for all $l \in \mathbb{X}'$, then X is Pettis integrable.*

Recall that a function is Bochner integrable iff it is both strongly measurable and in $L^1_{\mathbb{X}}$. In the context of a probability space, these conditions are satisfied by any measurable function with a precompact range.

Theorem 3.2. *Suppose X is a weakly measurable function from a probability space to a Banach space. If the range of X is precompact, then X is Bochner integrable.*

Suppose the underlying measurable space is a topological space with its Borel σ-algebra. If X is a continuous mapping, then precompactness of the range of X is implied by precompactness of its domain - see Exercise REF.

Look ahead to Exercise ?? for an existence fact when the codomain of f is a space of continuous real-valued functions.