Measurable spaces

introduction to MEASURE THEORY - mathematically formalizes the idea of the size of something being the sum of the sizes of its parts.

UNIFYING CONCEPT: “paving” for a type of class of subsets

1 Measurable spaces

1.1 Types of collections

In Section REF, we defined a topology as a collection of subsets of $\mathbb{X}$ that is closed under union and pair-wise intersection and contains the full set $\mathbb{X}$. Here we'll define a number of other types of collections of subsets, strictly increasing in structure.

A collection of sets $\mathbb{A}$ is called a semi-ring if it satisfies the following three properties:

(i) It is non-empty.
(ii) It is closed under pair-wise intersection.
(iii) Its set-differences are equal to finite disjoint unions of its sets.

1.1 Explain why a semi-ring must contain $\varnothing$.

1.2 Explain why a semi-ring must be closed under *finite* intersection.

1.3 Consider the axiom “It is closed under set-difference.” Explain why this condition is at least as strong as the second and third *semi-ring* axioms put together.

A collection of sets $\mathbb{A}$ is called a ring (SIDENOTE: A *ring of sets* is distinct from the *ring* concept we studied in Section REF.) if it satisfies the following three properties:

(i) It is non-empty.
(ii) It is closed under pair-wise union.
(iii) It is closed under set-difference.

Based on Exercise REF, we see that these axioms are stronger than the *semi-ring* axioms. Thus every ring must also be a semi-ring.

1.4 Explain why a ring of sets is also closed under symmetric difference and under finite intersection.

DEFINE SMALLEST RING $R(\mathbb{A})$ containing a collection a.k.a. the “ring generated by” a collection. (SIDENOTE: Why is this well-defined and unique? Any intersection of rings is a ring, so we can define the ring generated by $\mathbb{A}$ to be the intersection of all rings that are supersets of $\mathbb{A}$, if there are any. Are there? Yes, the power set of the universe is one. - I REALLY NEED to deal with this type of thing in a general way! Same with the closure under intersection property of all these types of collections.)

If $\mathbb{A}$ is a semi-ring, then $R(\mathbb{A})$ is exactly the collection of all finite unions of its sets. - SO SIMILAR to base and topology generate by base!

To define an algebra (or *field*) of sets, we replace the third axiom of *rings* with “It is closed under complement.”

1.5 Explain why a ring of subsets of $\mathbb{X}$ is also an algebra iff it contains $\mathbb{X}$. (SIDENOTE: This fact shows that every algebra is a ring, and what's more, we could have equivalently defined *algebra* by adding the axiom “It contains the universe” to the *ring* axioms. In fact, *semi-algebra* (or *semi-field*) is defined by adding the axiom “It contains the universe” to the semi-ring axioms.)

SMALLEST algebra is relevant to know about. notation: $\alpha(\mathbb{S})$. NEXT SECTION EXPLAINS: smallest algebra approximates smallest σ-algebra!

Finally, to define a σ-algebra (or σ-*field*), we replace the “closed under pair-wise union” axiom of *algebra* with the stronger requirement “closed under *countable* union.” That is, a collection of sets is a σ-*algebra* if it satisfies the following three conditions:

(i) It is non-empty.
(ii) It is closed under countable union.
(iii) It is closed under complement.

A set along with a σ-algebra $(\mathbb{X}, \mathbb{A})$ together are called a measurable space. Any subset $A \subseteq \mathbb{X}$ that is in $\mathbb{A}$ is called a measurable set (or an $\mathbb{A}$-*measurable* set if the σ-algebra isn't clear from context).

1.6 Show that σ-fields are also closed under countable intersection.

1.7 Explain why the intersection of any collection of σ-fields is itself a σ-field. (SIDENOTE: We've seen in Exercise REF that any intersection of topologies is itself a topology. This is also true for semi-rings, rings, and fields. The reasoning is basically the same in every case. ASK reader to formulate and prove a more general theorem that automatically applies to all of these cases!)

If $\mathbb{S}$ is a collection of subsets of $\mathbb{X}$, we let $\sigma(\mathbb{S})$ denote *the σ-field generated by* $\mathbb{S}$. It is defined to be the smallest σ-field containing $\mathbb{S}$. (SIDENOTE: How do we know that this is well-defined? There is at least one σ-field containing $\mathbb{S}$: the power set $2^{\mathbb{X}}$. In fact, the smallest σ-field containing $\mathbb{S}$ is exactly the intersection of all the σ-fields containing $\mathbb{S}$ (recall Exercise REF).) EXERCISE: If $\mathbb{A}$ is a semi-ring, then $\sigma(\mathbb{A}) = \sigma(R(\mathbb{A}))$

Consider as an example $\Omega = \mathbb{R}$ and the σ-field generated by $\mathbb{S}_0$, the collection all open intervals (a, b) with $-\infty < a < b < \infty$. Any σ-field containing the open intervals must also contain all the open sets in $\mathbb{R}$. (SIDENOTE: Recall from Section REF that every open set in $\mathbb{R}$ is a countable union of open intervals. WHY COUNTABLE? - every open set in a separable metric space is a countable union of open balls. MAKE a point of this somewhere in that chapter.) For more general types of Ω (specifically, topological spaces), we define the Borel σ-field, denoted $\mathbb{B}(\Omega)$, to be the σ-field generated by the set of all open subsets of Ω. The sets in a Borel σ-field are called the Borel sets.

The Borel σ-field of $\mathbb{R}$ is of central importance in measure theory. It is by definition the σ-field generated by the open subsets $\mathbb{S}$, but it is also generated by $\mathbb{S}_0 \subset \mathbb{S}$, the collection of open intervals, a fact that follows from the discussion in the previous paragraph. Indeed there are a number of other useful choices of generating class for $\mathbb{B}(\mathbb{R})$, including the collection $\{[t, \infty) : t \in \mathbb{R}\}$. Before attempting to show this, we will introduce a general method of showing that two collections generate the same σ-field.

To show that $\sigma(\mathbb{S}_1) = \sigma(\mathbb{S}_2)$, one can show inclusion in both directions. It is sufficient to show that $\mathbb{S}_1 \subseteq \sigma(\mathbb{S}_2)$ and $\mathbb{S}_2 \subseteq \sigma(\mathbb{S}_1)$, because the σ operator returns *the intersection of all σ-fields* containing its argument. (SIDENOTE: This reasoning may be a bit tricky when you first see it, so let's think carefully. Assume you have shown that $\mathbb{S}_1 \subseteq \sigma(\mathbb{S}_2)$. By definition, we also know that $\sigma(\mathbb{S}_1)$ is the intersection of all σ-fields that contain $\mathbb{S}_1$. One of those σ-fields being intersected is $\sigma(\mathbb{S}_2)$, so it must be a superset of $\sigma(\mathbb{S}_1)$.) First, pick an arbitrary element of $\mathbb{S}_1$ and show that it can be constructed by doing sigma-field-preserving operations to elements from $\mathbb{S}_2$. Then vice versa.

1.8 Using the method described in the preceding paragraph, prove that $\mathbb{S}_1 := \{[t, \infty) : t \in \mathbb{R}\}$ generates $\mathbb{B}(\mathbb{R})$.

1.9 Are the singletons $\{t\}$ in $\mathbb{B}(\mathbb{R})$.

(SIDENOTE: One might consider using structural induction (see Section REF) to extend properties from a generating class $\mathbb{S}$ to its σ-field. Notice, however, that $\sigma(\mathbb{S})$ *was not defined by the recursive action* of taking complements and countable unions; we only know that it contains all of its complements and countable unions. So structural induction could extend a property to all of the complements and countable unions of sets in $\mathbb{S}$, but there might be other members in $\sigma(\mathbb{S})$ that it misses.)

COUNTABLY-GENERATED sigma field (or measure space) - compare to second-countable topology!

EXERCISE: Show that if a topological space is second-countable, then its Borel σ-algebra is countably-generated. (AND REMIND reader that for metric spaces, second-countability is equivalent to separability!)

1.2 Borel σ-algebras

If $\mathcal{T}$ is a topological space with topology τ, then we call $\sigma(\tau)$ the Borel σ-algebra of $\mathcal{T}$. In particular, if $\mathcal{T}$ is a Polish space, then the measurable space $(\mathcal{T}, \mathbb{B}(\mathcal{T}))$ is called a standard Borel space. For most of the common applications of measure theory, the underlying measure space is a standard Borel space; conveniently, however, there “aren't many” standard Borel spaces.

Proposition 1.1. *Two standard Borel spaces are isomorphic iff they have the same cardinality.*

Propositions REF and REF together tell us that all uncountable standard Borel spaces are isomorphic to each other and have the cardinality of the continuum; conveniently, this reduces questions about standard Borel spaces to questions about the interval $[0, 1]$ with its Borel σ-algebra.

Proposition 1.2. *The restriction of a standard Borel space to any measurable subset is itself a standard Borel space.*

Proposition 1.3. *The product of countably many standard Borel spaces is itself a standard Borel space.*

measurable functions:

A bijective function from one standard Borel space to another is measurable iff its inverse is measurable.

product-measurability of functions:

A function from one standard Borel space to another is measurable iff its graph is measurable.

1.3 Product spaces

Consider two measurable spaces $(\Omega_1, \mathbb{A}_1)$ and $(\Omega_2, \mathbb{A}_2)$. We define the *product* of the two σ-fields by

$$\mathbb{A}_1 \otimes \mathbb{A}_2 := \{A_1 \times A_2 : A_1 \in \mathbb{A}_1, A_2 \in \mathbb{A}_2\}$$

which is a collection of subsets of $\Omega_1 \times \Omega_2$. (SIDENOTE: Notice that this operation isn't quite the same thing as Cartesian product. EXPLAIN.) The product σ-field is the σ-field generated by $\mathbb{A}_1 \otimes \mathbb{A}_2$. This concept extends to arbitrary collections of measurable spaces.

MEASURABLE RECTANGLES comprise a semi-ring - MAKE EXERCISE:

ALSO DEFINE product topology when the sets have topologies. SOMETIMES this is what is meant by “product space” I think - depends on context.

IT IS POSSIBLE to define a set in $\mathbb{X} \times \mathbb{Y}$ that is measurable for each space individually but not product measurable (MAKE IT AN EXERCISE to devise such a set - my solution could go over the class Lebesgue example. also there's a neat countable/uncountable diagonal example i saw on stackexchange that applies in more general settings). IS THAT still possible if we're willing to extend to the completion of the product space?

ABOUT completion of $\mathbb{A} \otimes \mathbb{B}$? IT's not the case that the completion of $\mathbb{A} \otimes \mathbb{B}$ is equal to the product space of the completions! (e.g. the Lebesgue measure on $\mathbb{R}^2$ is not the product of two copies of Lebesgue measure on real lines.)

2 Measurable functions

The term *measurable* is also applied to functions. Let f be a function from one measurable space to another ($f : \Omega_1 \to \Omega_2$), with σ-fields $\mathbb{A}_1$ and $\mathbb{A}_2$. We say that f is an $\mathbb{A}_1/\mathbb{A}_2$-measurable function if the pre-image of each $A \in \mathbb{A}_2$ is a set in $\mathbb{A}_1$, (SIDENOTE: One or both of the σ-fields may be omitted from the term if they are clear from context, e.g. “$\mathbb{A}_1$-measurable function” or just “measurable function.”) i.e. for all $A \in \mathbb{A}_2$,

$$f^{-1}A := \{\omega \in \Omega_1 : f(\omega) \in A\} \in \mathbb{A}_1$$

Checking that a function is measurable may seem impossible. Fortunately, we don't actually have to verify the condition for every subset in the codomain's σ-field. If the condition is satisfied for all subsets of a generating class for that σ-field, then it is guaranteed to hold for the whole σ-field. This is a consequence of the following exercise.

4. Show that the set of $A \in \mathbb{A}_2$ for which $f^{-1}A \in \mathbb{A}_1$ comprises a σ-field.

VECTOR SPACE of measurable functions: Notation e.g. $\mathbb{M}_{(\mathbb{X},\mathbb{A})}(\Omega, \Sigma)$. Short-hand, depending on what should be clear from context. e.g. $\mathbb{M}_{\mathbb{X}}$ - if the subscript is omitted, the assumption is $\mathbb{R}$ with its Borel-sigma field - or is it $\bar{\mathbb{R}}$? Check pollard's book.

We define $\sigma(f)$ to be the smallest σ-field on $\mathbb{A}_1$ for which f is measurable. It is called the *sigma field generated by* f. Crucially, it has an interpretation in terms of how informative f is about Ω. If you were to learn the value of $f(\omega_0)$, the σ-field $\sigma(f)$ is precisely the collection of subsets of Ω for which you would be able to definitively say whether or not ω_0 is in that subset. The more f “distinguishes” the sample space elements (by mapping them to different outputs), the more refined $\sigma(f)$ is. For instance, if f is injective, then $\sigma(f)$ is the power set of Ω.

ABOUT a function f being *g-measurable* - it's σ-field is a sub-sigma-field of g's - then it can be expressed as a deterministic function of the output of g. IT can't tell us anything that g doesn't - if you're trying to determine the source ω_0 based on the observed function output $f(\omega_0)$ and $g(\omega_0)$, you don't lose anything by ignoring the f output in favor of the g output. what if the subset it proper $\sigma(f) \subset \sigma(g)$? What exactly can you say about the “inferiority” of f?

There's a nice notation in “Some notes on standard Borel and related spaces” by Chris Preston - f is measurable if $f^{-1}(\mathbb{A}_2) \subseteq \mathbb{A}_1$.

MAYBE HAVE A SECTION on the nature of the “sample space.” - i think this is relevant when checking whether L^p spaces are separable.

A measurable space is called separable (SIDENOTE: This concept is distinct from *separability* of topological spaces. BUT IT'S apparently related via the measure metric - maybe its equivalent or almost equivalent modulo null sets or something) if the σ-field includes every singleton. (SIDENOTE: If it contains all singletons and is closed under countable union, must the σ-field be the power set? Not necessarily. Every countable set must be in this σ-field, but uncountable sets don't have to be.)

A measurable space is countably generated if it has a countable generating class.

A metric space is separable iff its topology has a countable base. Then USE PRESTON Prop 3.1 - the Borel space of a metric space is countably generated as well - THIS is a good exercise I think.

A countably generated measurable space has a countable algebra. (Preston Prop 3.3) THE LINK below seems to indicated a stronger result: the algebra generated by a countable set is itself countable.

measurability of maxima and suprema

EXERCISE: Does $\mathbb{B}(\mathbb{R})$ contain the singletons? What about $\mathbb{B}(\bar{\mathbb{R}})$?

Limit of measurable functions is measurable.

“measure algebra” might be a useful term/concept for this book - would need to be introduced in the measures document.

include a $\mathbb{B}(\bar{\mathbb{R}})$ discussion somewhere - is it understood as an actual Borel σ-algebra? What is the “natural” topology on $\bar{\mathbb{R}}$ supposed to be?

EXERCISE: A function mapping to a metric space $\mathbb{M}$ is called *Price measurable* if the pre-image of every open ball is measurable. Prove that if $\mathbb{M}$ is separable, then Price measurability is equivalent to Borel measurability.

2.1 Product-measurable functions

EXPLAIN product-measurability and state that it will be really important.

EXERCISE: Explain why a $(\bigotimes_i \mathbb{A}_i)$-measurable function is also product-measurable as a mapping from a subset of the $(\mathbb{X}_i, \mathbb{A}_i)$ when the remaining variables are held fixed.

It would be convenient if we had some simple sufficient conditions to check for product measurability, and in many cases this is indeed possible. If $(\mathbb{X}, \mathbb{A})$ is a measurable space and $\mathbb{Y}$ and $\mathbb{Z}$ are topological spaces, then a Carathéodory function is a mapping from $\mathbb{X} \times \mathbb{Y}$ to $\mathbb{Z}$ that is $\mathbb{A}/\mathbb{B}(\mathbb{Z})$-measurable for every fixed $y \in \mathbb{Y}$ and continuous for every fixed $x \in \mathbb{X}$.

1.10 Prove that if f is a Carathéodory function with $\mathbb{Y}$ separable and metrizable and $\mathbb{Z}$ metrizable, then f is $(\mathbb{A} \otimes \mathbb{B}(\mathbb{Y}))/\mathbb{B}(\mathbb{Z})$-measurable.

DO I REALLY want to make this an exercise rather than a skyhook theorem?

SOLUTION: Carathéodory functions EXERCISE - THE PROOF and more results along these lines are in the book “Infinite dimensional analysis: a hitchhiker's guide” section 4.10.

FOR THE CARATHEODORY functions thing ... I think this will be a common scenario: maybe the domain $\mathbb{X}$ isn't necessarily separable, but the range of $x \mapsto \phi_x$ is essentially separable (does that happen when $\mathbb{X}$ is a Polish space?) - is that good enough? I WOULD THINK SO! JUST restrict the domain to the subset.

APPARENTLY right-continuous is also sufficient in the case of $\mathbb{X} \times \mathbb{R} \mapsto \mathbb{R}$. THAT could be very useful to know as well, because it gives us CDFs!

HERE's something where the codomain is an L^p space. Could be useful to me - but this might be redundant based on the Caratheodory exercise.

ANOTHER good EXERCISE along these lines:

MOVE THIS TO A PRODUCT measurability section of the measure theory document.

DEMONSTRATE that measurability in each variable doesn't imply product measurability!

However, in some cases one can verify that a function is product-measurable just be considering its behavior on each variable. A function from $(\Omega, \Sigma) \times (\mathbb{V}, \mathbb{B}(\mathbb{V})$ to a TVS is called a Carathéodory function if it is continuous in x for each fixed ω and Borel measurable in ω for each fixed x. When the spaces involved have the right structure, Caratheodory functions are product-measurable.

Theorem 2.1. *Let $\mathbb{X}$ be a separable Banach space and $\mathbb{Y}$ be a Banach space. Suppose that for each ω in a set Ω, the function $f_\omega : \mathbb{X} \to \mathbb{Y}$ is continuous. Let Σ be the σ-algebra on Ω generated by the set of functions $\{\omega \mapsto f_\omega(x) : x \in \mathbb{X}\}$. The mapping $(\omega, x) \mapsto f_\omega(x)$ is $\Sigma \otimes \mathbb{B}(\mathbb{X})$-measurable.*

CITE infinite dimensional analysis: a hitchhiker's guide LEMMA 4.51

2.2 Measurability of TVS-valued functions

setting: f maps from (Ω, Σ) to a TVS - the Borel σ-algebra is assumed to be the one of interest.

DO All Borel σ-fields include the singletons? true for $\mathbb{R}$ - in general, how much structure is required for it to hold?

Weak measurability

f is weakly measurable if $l(f)$ is Borel measurable for every bounded linear functional l in the dual space $\mathbb{V}'$.

EXERCISE: Explain why every Borel measurable function is also weakly measurable.

Strong measurability

A simple function from (Ω, Σ) to $\mathbb{V}$ has the form

$$s(\omega) = \sum_{i=1}^{n} \mathbb{I}_{A_i}(\omega) v_i$$

where each A_i is in Σ and each v_i is in $\mathbb{V}$. (SIDENOTE: It is easy to see that the simple functions are exactly the set of measurable $\mathbb{V}$-valued functions that take on only finitely many values.) The set $\mathbb{S}_{\mathbb{V}}(\Omega, \Sigma)$ of simple functions is a linear subspace of the vector space of all functions from Ω to $\mathbb{V}$.

EXERCISE: Explain why every simple function is Borel-measurable. EXERCISE: Suppose a simple function s can be represented with n terms. State an upper bound on the cardinality of the range of s. SOLUTION: 2^n.

SPACE of all simple functions from (Ω, Σ) to $\mathbb{X}$ is called $\mathbb{S}_{\mathbb{X}}(\Omega, \Sigma)$ OMITTING subscript (i.e. “$\mathbb{S}$”) implies the codomain is $\mathbb{R}$. ALSO define $\mathbb{S}^+$ to be the non-negative simple functions - it's the same set of functions you would get by insisting that you could only use non-negative reals in your representations.

A function f that maps from (Ω, Σ) to a Banach space is called strongly measurable if it can be approximated by simple functions. In this context, we mean that there exists a sequence (s_n) of simple functions with a point-wise limit of f, i.e. for all $\omega \in \Omega$

$$\lim_{n \to \infty} \|f(\omega) - s_n(\omega)\| = 0. \tag{1}$$

As the names indicate, strong measurability is *stronger* than Borel measurability, which is itself stronger than weak measurability. In fact, the relationship is stated more precisely by the following theorem.

Theorem 2.2 (Pettis Measurability Theorem)**.** *Let f be a function from (Ω, Σ) to a Banach space $\mathbb{X}$. The following are equivalent:*

- *f is strongly measurable.*
- *f is Borel measurable and essentially separably valued.*
- *f is weakly measurable and essentially separably valued.*

Notice that if $\mathbb{X}$ is separable, then these three notions of measurability are all equivalent.

NOTATION: $M_{\mathbb{X}}(\Omega, \Sigma)$ for the set of all strongly measurable functions.

EXTEND strong measurability to $\bar{\mathbb{R}}$-valued functions. EXPLAIN. WHEN WE OMIT the subscript, i.e. “$M(\Omega, \Sigma)$,” the codomain is understood to be $\bar{\mathbb{R}}$. AND $M^+(\Omega, \Sigma)$ is the set of non-negative functions in $M(\Omega, \Sigma)$.

- MAKE AN EXERCISE relating ordinary $\mathbb{B}(\bar{\mathbb{R}})$-measurability to strong-measurability.

EXERCISE: Show that the set of strongly measurable functions is a subspace. Show that the set of weakly measurable functions is a subspace. (Gasinki Remark 2.1.6. for strongly measurable, use triangle inequality. for weakly measurable - easily follows from the fact that ordinary measurability is preserved under linearity? or easier proof?) MAYBE repeat for convex cones.

EXERCISE: Explain why a composition of measurable functions is itself measurable.

EXERCISE: Let f be a function from (Ω, Σ) to a topological vector space $\mathbb{Y}$. Prove that if f is $\Sigma/\mathbb{B}(\mathbb{Y})$-measurable, then it is also weakly measurable. SOLUTION ... - bounded linear functionals are continuous, therefore Borel-measurable - their compositions with f must also be measurable.

EXERCISE: Show that the point-wise limit of a sequence of strongly measurable functions is strongly measurable.

COR 1.7: If f is an $\mathbb{X}$-valued strongly measurable function and T is a continuous linear operator mapping from $\mathbb{X}$ to a Banach space $\mathbb{Y}$, then Tf is also strongly measurable. (THESE DO NOT SUBSUME the analogous results for ordinary measurability though - those are still informative for functions that are measurable without necessarily being strongly measurable.)

IN PARTICULAR, clarify the relationship between strong measurability and Borel-measurability on $\bar{\mathbb{R}}$ - i.e. they are equivalent! - this will be crucial to understanding that my construction is equivalent to the usual ones.

LEAD IN TO NEXT SECTION: Now that we've identified various objects as “measurable,” let's see what it means to “measure” them...