POINT TO relevant Section in Chapter 0 where definition of vector space, operator, functional, etc. can be found.

1 Norms and normed spaces

Define norm as a notion of difference that requires more structure than metric. Define normed space.
POINT out that field is a general notion, but we will specialize to the field real numbers for our discussion. *In this book, all vector spaces will use real numbers as their fields unless otherwise specified.*
Let V be a real vector space. A norm (denoted by $\|\cdot\|$) is a real-valued functional on V satisfying the following conditions for any $x, y \in V$ and any $\alpha \in \mathbb{R}$.

- **Absolute homogeneity:** $\|\alpha x\| = |\alpha|\|x\|$
- **Separates points:** $\|x\| = 0$ implies x is the zero vector (SIDENOTE: To be consistent with our terminology in Section REF, the "separating points" property should have a little more to it. Namely, $\|x\| \geq 0$ with equality if $x = 0$. This is actually implied by the other two norm properties; see Exercises REF and REF. So know that norms do indeed have the *true* separating points property, but we only need to take a piece of that property as a defining axiom.)
- **Triangle inequality:** $\|x+y\| \leq \|x\| + \|y\|$ (SIDENOTE: For functionals in general, this property is called subadditivity. I THINK THIS will be defined in the induction section of document 0-1.)

Any vector space taken together with an associated norm is called a normed vector space.

1. Show that any functional f with absolute homogeneity must have $f(0) = 0$.
2. Show that any subadditive functional f with absolute homogeneity must be non-negative.
3. Show that any norm defines a metric on the vector space by $d(x, y) := \|x - y\|$. We will call any such d a norm distance.
4. Show that any norm distance is convex in either of its arguments.
5. Show that $\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$.
6. Suppose f is a functional on V that satisfies the definition of norm, except that it takes infinite values for some of the vectors. Let $V_0 \subset V$ be the set of all vectors for which f is finite. Show that V_0 is a vector subspace of V. Assume a sequence of vectors in V (each with finite f values) converges to a limiting vector. Show that the limiting vector must also have a finite f value.

We saw in Exercise NUM that all norms define metrics. But it's not the case that all metrics correspond to a norm. (SIDENOTE: For example, recall the *discrete space* from REF; that metric doesn't scale like a norm is supposed to. Furthermore, the underlying set of a metric space might not even be a vector space, which is required for a norm to make sense.) Recall the discrete metric, for example. And if the) It's easy to verify that the two following conditions on such a metric, taken together, are necessary and sufficient for it to correspond to a norm:

- translation invariance: $d(x, y) = d(x + a, y + a)$
- absolute homogeneity: $d(\alpha x, \alpha y) = |\alpha| d(x, y)$

The norm distance is $d(x, y) = \|x - y\|$, so the resulting norm is $\|x\| = d(x, 0)$, the distance from zero.

EXERCISE: the weird sequence space (on all sequence) from the previous section doesn't define a norm. MAKE it an exercise to show that it doesn't satisfy one (or both?) of the criteria.

EXERCISE: THE set of functions from $\mathcal{X}$ to $\mathcal{Y}$ is a vector space. use the TWO CONDITIONS to show that supremum metric comes from a norm (and find the norm!) THE norm is $\|x\| := \sup_{a \in S} |x(a)|$. SIDENOTE: ALSO, (NOW that you know about normed spaces) this metric (and norm) could be generalized for functions mapping to normed spaces by replacing the absolute value with a norm.

EXERCISE: Recall that integral transforms (REF SECTION) are linear operators. (NEED any caveats on the linearity?) Let T be the integral transform corresponding to some kernel on $\Theta \times \mathcal{X}$. Let s denote the supremum of the kernel on $\Theta \times \mathcal{X}$, and assume that it is finite. The domain of the transform is the space of all densities of signed measures on Θ. With the supremum norm on the domain and codomain, show that T is a *bounded* linear operator. (KREY 2.7-6)

Consider two norms on a vector space V. If there exists positive numbers $a \leq b$ such that for every $x \in V$,

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

then the norms are called *equivalent*. Equivalent norms define equivalent metrics and thus induce the same topology. It can be shown that all norms on a finite-dimensional vector space are equivalent. (Krey 2.4-5)

Another way that finite-dimensional normed spaces are simpler relates to compactness. Recall that every compact subset of a metric space must be closed and bounded. The converse is not true in general, but it is true for finite-dimensional normed spaces. So in a finite-dimensional normed space, the collection of compact subsets is exactly the collection of closed and bounded subsets. (Krey 2.5-3)

INFINITE SUMS - distinguish conditionally convergent, unconditionally convergent, absolutely convergent. (DOES this go in the Banach spaces section or before it? i.e. is completeness relevant?)

2 Banach spaces

A complete normed space is called a Banach space. All finite-dimensional normed spaces are complete (Krey 2.4-2), but infinite-dimensional normed spaces may not be.
As with metric spaces, it can be shown that any normed space V can be "filled in" to create a unique *completion*, a Banach space for which V is dense. (Krey 2.3-2)

7. Explain why a finite-dimensional subspace X of a normed space Y has to be closed in Y.

Because normed spaces are vector spaces, every normed space has a Hamel basis, as defined in Section REF. (SIDENOTE: Previously we just called this a *basis*. But in the context of infinite-dimensional normed spaces, one should use the full term "Hamel basis" to distinguish it from another type of "basis" that you will learn about in Section 2.1.) All Hamel bases have the same cardinality, which is called the *dimension* of the vector space. Strangely, it can be shown that a Banach space cannot have a countably infinite Hamel basis. Even stronger, all infinite-dimensional Banach spaces have dimension at least $\aleph_1$. (SIDENOTE: This is only stronger if the continuum hypothesis is untrue; otherwise it is equivalent.)

2.1 Schauder basis

We can generalize the *series* concept from numbers to vector spaces. Given any sequence of vectors, the corresponding series is the sequence of partial sums. For normed spaces, we say the series is *convergent* if it approaches a limiting vector. It is *absolutely convergent* if the norms of the sequence vectors comprise a convergent series. (SIDENOTE: If the normed space isn't complete, then a series can be absolutely convergent without also being convergent. For Banach spaces, a absolutely convergent series must also be convergent, but convergent series might not be absolutely convergent - the terms themselves don't have to get small as long as they counteract each other enough.)

With normed vector spaces, we have the possibility of a sequence of vectors *approximating* some desired vector. One way to try approximation is to continue adding terms into a linear combination; perhaps the resulting infinite series might even converge to the desired vector. Making use of this idea, we define another type of "basis" for normed spaces. (SIDENOTE: Review the definition of [Hamel] basis to see the contrast with this new type of "basis.") A Schauder basis (or sometimes "countable basis," a painfully misleading term) for V is a sequence of vectors for which any $x \in V$ can be uniquely expressed as an infinite summation over scalar multiples of the Schauder basis vectors. The sum representation of x is called its *expansion* with respect to that Schauder basis.

We can say a few things about the "size" of a normed space that has a Schauder basis. First, it must be infinite-dimensional (because its elements need to be linearly independent of each other). (SIDENOTE: As we mentioned before, infinite-dimensional Banach spaces must have uncountable Hamel bases. But if such a space also has a Schauder basis, the countability of the Schauder basis makes it more convenient in many ways.) Any normed space with a Schauder basis must also be separable. It was long conjectured that the converse was true for Banach spaces: separable implies Schauder basis. Although most of the commonly studied separable Banach spaces do have a Schauder basis, it is possible to devise a separable Banach space that does not have one. (CITE Enflo 1973 - see Krey pg 69)

EXERCISE: Prove that any normed space with a Schauder basis must be separable. (Krey 2.3 Prob 10)

8. Let B be a Banach space with a countable basis $b := (b_1, b_2, \ldots)$. The coordinate functional π_i maps any vector $v \in B$ to the coefficient of b_i in the b-expansion of v. Prove that the coordinate functional is linear and continuous.

THIS DOCUMENT SHOULD START BY COVERING topological vector spaces! THEN move on to LOCALLY CONVEX HAUSDORFF and in particular the Hahn-Banach Thm - finally add structure to get to normed spaces. - don't dwell on LCHS because the definition is complicated - the most important thing is that they are all normed spaces. they have a slightly weaker definition but it is strong enough (just barely strong enough?) to get Hahn-Banach.
BEWARE: different authors use different definitions of TVS. e.g. Definition 3.9 in the document below includes the requirement that every singleton be a closed set - it turns out that this automatically implies that the TVS is Hausdorff (Corollary 3.7).
OR SOME authors include Hausdorff in the definition of locally convex spaces, apparently. I don't want to do that - I want to be more explicit, and use the name locally convex Hausdorff space (LCHS) for the spaces that have Hahn-Banach and are therefore of most interest in the context of Pettis integrals.

Solution

1. Substituting $\alpha = 0$ into the definition of absolute homogeneity gives, for any vector v,
$$\begin{aligned} f(0v) &= |0| f(v) \\ \Rightarrow f(0) &= 0 \end{aligned}$$
Because norms by definition have absolute homogeneity, we see that $\|0\| = 0$.
2. For this one, we can make use of the observation we made in Exercise (NUMBER). Let v be any vector in the vector space.
$$\begin{aligned} \|v\| &= \frac{\|v\|}{2} + \frac{\|v\|}{2} \\ &= \|v/2\| + \|v/2\| \\ &\geq \|v/2 - v/2\| \\ &= \|0\| \\ &= 0 \end{aligned}$$
Because norms are by definition subadditive and have absolute homogeneity, this result applies, telling us that norms must be non-negative.
3. SHOW that norm distance $d(x, y) := \|x - y\|$ satisfies the definition of metric. I STILL NEED TO TYPE THIS UP - BUT IT'S EASY!

   To go back from the distance to the original norm, notice that $\|x\| = d(x, 0)$. This shows us that norms inherit metric properties. For instance, the norm is a continuous functional.
4. We only need to show convexity in one argument. Convexity in the other argument follows by symmetry.
$$\begin{aligned} d(\lambda x_1 + [1-\lambda]x_2, y) &:= \|(\lambda x_1 + [1-\lambda]x_2) - y\| \\ &= \|\lambda x_1 - \lambda y + [1-\lambda]x_2 - [1-\lambda]y\| \\ &\leq \|\lambda x_1 - \lambda y\| + \|[1-\lambda]x_2 - [1-\lambda]y\| \\ &= \lambda\|x_1 - y\| + [1-\lambda]\|x_2 - [1-\lambda]y\| \\ &= \lambda d(x_1, y) + [1-\lambda] d(x_2, y) \end{aligned}$$
It is good to know when a given metric comes from a norm, because then some nice properties such as convexity come along automatically.
5. We will use the triangle inequality and Young's inequality.
$$\begin{aligned} \|x+y\|^2 &\leq (\|x\| + \|y\|)^2 \\ &= \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &\leq \|x\|^2 + \|y\|^2 + 2\left(\tfrac{\|x\|^2}{2} + \tfrac{\|y\|^2}{2}\right) \\ &= 2(\|x\|^2 + \|y\|^2) \end{aligned}$$
6. Triangle inequality and absolute homogeneity let us show that $x, y \in V_0$ implies $ax + by \in V_0$.
$$\begin{aligned} f(ax+by) &\leq f(ax) + f(by) \\ &= af(x) + bf(y) \\ &< \infty \end{aligned}$$
So if we ever identify a functional with norm-like properties that can take infinite values, we can always define a normed space by the vectors for which it is finite. With this idea in mind, we may use the norm notation even with it might produce infinite values; the understanding is that any vectors with infinite values aren't part of the actual *normed space.*

   Use the triangle inequality for the limiting vector and some sequence vector. (For this part, we'll use the norm notation for f with the understanding that we can't take its finiteness for granted.)
$$\begin{aligned} \|x\| &= \|x - x_n + x_n\| \qquad \text{true for } \textit{any given } n \\ &\leq \|x - x_n\| + \|x_n\| \end{aligned}$$
Because the first term is converging to zero, there is certainly some N for which $\|x - x_N\|$ is finite. And based on the problem's assumption, $\|x_N\|$ is finite as well.

   (reasoning based on Pollard page 39)
7. We know that X is also a subspace of $\bar{Y}$, the Banach space that is the completion of Y. By Theorem REF (Krey 1.4-7 - make sure I've got it as a theorem too), X must be closed in $\bar{Y}$. Any set closed in a space must also be closed in all subspaces, so X is closed in $Y \subseteq \bar{Y}$.

   Krey Thm 2.4-3
8. Linearity is easy. Let $c, d \in \mathbb{R}$ and $u, v \in B$. Then
$$\begin{aligned} cu + dv &= c\sum_j u_j b_j + d\sum_j v_j b_j \\ &= \sum_j (cu_j + dv_j) b_j \end{aligned}$$
This representation tells us exactly what resulting coefficients are. So $\pi_i(cu + dv) = cu_i + dv_i = c\pi_i(u) + d\pi_i(v)$.

   The boundedness of the coordinate functional is harder to show. If we were only looking at finite linear combinations, then there is a straight-forward proof based on (CITE KREYSZIG, Lemma 2.4-1).

   GO THROUGH THE PROOF and cite StackExchange

   DOES THIS hold for topological vector spaces in general? If so, make that the question instead!