Topological Spaces

We begin this chapter with a glimpse at a type of space with very little structure. It turns out that many familiar concepts about subsets of real numbers continue to make sense in a much more abstract setting. And in many cases, their properties and relationships are actually *more straightforward* once we've stripped everything down to the bare essentials.

- LEADS to convenience in dealing with [divergence and] metric spaces.

- REMARKABLE similarities with aspects of measure theory (compare topologies to sigma fields and continuity with measurability).

1 Topologies

MOST OF THIS STUFF SHOULD GO IN chapter 0-1

Given a set Ω, a collection $\mathbb{T}$ of subsets is called a topology for Ω if it satisfies the following three conditions.

- It contains the full set Ω.
- It is closed under union. (SIDENOTE: Realize that this includes *uncountable* unions. It also includes the *trivial* union of no sets, which results in the empty set.)
- It is closed under pair-wise intersection. (SIDENOTE: By induction, it follows that a topology is also closed under finite intersection.)

The pair $(\Omega, \mathbb{T})$ is called a topological space. (SIDENOTE: Any set "automatically" has two topologies. One is $\{\varnothing, \Omega\}$, which is called the trivial topology. Another is the power set 2^Ω, which is also called the discrete topology.) The elements of Ω are called points, and the sets in $\mathbb{T}$ are called the open sets. (SIDENOTE: For the moment, don't rely too much on your familiar intuitions about what "open" means in the context of, for instance, real intervals. (Inspired by that intuition, we're setting out to establish a body of much more general results.) In topological spaces, a set being open *just means* that it is in the topology. Although, we'll see that *when additional structure is present* this "openness" typically does coincide with your intuition.)

1. Is an intersection of topologies also a topology? What about a union of topologies?
2. A pair $(\Omega_0, \mathbb{T}_0)$ is a topological subspace of the topological space $(\Omega, \mathbb{T})$ if $\Omega_0 \subseteq \Omega$ and the sets in $\mathbb{T}_0$ are defined to be the intersections of Ω_0 with the sets in $\mathbb{T}$. Explain why a topological subspace is a topological space.

It might be helpful for the reader to review Section REF for the definition of *topology* along with a few properties it has in common with other pavings.

2 Topological spaces

A topological space is a set Ω along with a topology of subsets $\mathcal{T}$. Here we will define a barrage of terms that categorize the points and sets of $(\Omega, \mathcal{T})$.

A neighborhood of $x \in \Omega$ is any set that *contains an open set containing* x. The neighborhood system of x (denoted $\mathbb{N}(x)$) is the collection of all its neighborhoods. (SIDENOTE: Analogously, one can also refer to a *neighborhood* or the *neighborhood system* of a *set*.)

A set is closed if its complement is open. It follows that a set is open if its complement is closed.

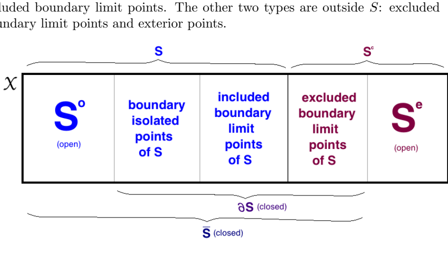
3. Show that an intersection of closed sets is closed.

Given a set S, there are a number of ways of categorizing the points in Ω with regard to their relationship with S. The interior of S (denoted S^o) is the *union of all open subsets* of S. The closure of S (denoted $\bar{S}$) is the *intersection of all closed supersets* of S. From the definitions, it is clear that $S^o \subseteq S \subseteq \bar{S}$. The boundary of S (denoted ∂S) is the symmetric difference $\bar{S} \backslash S^o$. The exterior of S (denoted S^e) is the complement of $\bar{S}$. Together, any set's interior, boundary, and exterior *partition* the space Ω. (SIDENOTE: How do we know they're a partition? Because the boundary is defined in a way that excludes the interior. Then the exterior is defined as everything that's left over.)

4. Show that $x \in S^o$ iff S contains a neighborhood of x.
5. Show that $x \in S^e$ iff S^c contains a neighborhood of x.

The boundary is the complement of $S^o \cup S^e$. In light of Exercises REF and REF, we can conclude that $x \in \partial S$ iff every neighborhood of x has *at least one point from each of S and S^c*. It is worthwhile to further subdivide the boundary into *boundary isolated points* and two types of *boundary limit points*. We call $x \in S$ a boundary isolated point of S if every neighborhood of x contains at least one point in S^c and there exists a neighborhood of x that has no other points of S. We call $x \in \mathbb{S}$ an included boundary limit point of S if every neighborhood of x contains at least one point in S^c and at least one point in S other than x itself. We call $x \in S^c$ an excluded boundary limit point of S if every neighborhood of x contains at least one point in S.

Thus, given a set S, we've described a way of neatly partitioning Ω into five types of points based on various relationships with S. (SIDENOTE: To convince yourself that this is a partition, look carefully at all the definitions, using the characterizations of interior and exterior from Exercises REF and REF.) Three types are inside S: interior points, boundary isolated points, and included boundary limit points. The other two types are outside S: excluded boundary limit points and exterior points.



6. Explain why S^o is the *largest open subset of* S and why $\bar{S}$ is the *smallest closed set containing* S.
7. Justify the following alternative characterizations of open and closed: S is open iff $S = S^o$; S is closed iff $S = \bar{S}$.
8. How do we know that ∂S is closed?

In light of Exercise REF and our partitioning scheme, we have a clear and detailed characterization of open and closed sets. A set is open iff it has no boundary isolated points or included boundary limit points; a set is closed iff it has no excluded boundary limit points.

9. We call $x \in S$ an isolated point of S if there exists a neighborhood of x that has no other points of S. We've seen points of this type in the boundary. When is an *interior* point isolated?
10. We call x a limit point of S if every neighborhood of x contains at least one point in S other than x itself (which may or may not be in S). (SIDENOTE: Notice that the interior can be partitioned into *interior isolated points* and *interior limit points*.) Explain why the limit points are the union of the interior limit points and the boundary limit points.
11. By our definition of topology, the full set Ω is open. Is it closed?
12. A set S is called dense in another set S' if $S' \subseteq \bar{S}$. Show that S is dense in the full set Ω iff every non-empty open set contains at least one member of S.

A net in Ω converges to a point $x \in \Omega$ if for neighborhood $M \in \mathbb{N}(x)$, there exists an α_M such that $\alpha \geq \alpha_M$ implies $x_n \in M$. This x is called the limit of (x_n), and familiar limit notation is used in this context.

13. Explain why any net with finite cardinality has a limit.
14. Prove that a point is in the closure of S iff it is the limit of a net in S.
15. Show that if there exists a sequence in S that converges to x, then x must be in $\bar{S}$. Devise an example showing that a point can be in the closure while no sequence converges to it. Thus in general, the relationship between convergence and closure is not quite as strong if you're only considering sequences (i.e. nets with index set $\mathbb{N}$). However, we'll see in Exercise REF a regularity condition on topological spaces that guarantees that an analogue Exercise 14 holds for sequences.
16. Show that a net converges to a point iff all of its subnets converge to that point.
17. A point x in a net $\{x_\alpha\}$ is called a net limit point if for every for neighborhood $M \in \mathbb{N}(x)$ and every index α there exists some $\beta \geq \alpha$ such that $x_\beta \in M$. Show that x is a net limit point of $\{x_\alpha\}$ iff it is the limit of some subnet of $\{x_\alpha\}$.
18. If the topological space is also a complete lattice, then we define the limit inferior (also called *liminf*) and limit superior (*limsup*) by
$$\liminf x_\alpha := \sup_\alpha \inf_{\beta \geq \alpha} x_\beta \qquad \text{and} \qquad \limsup x_\alpha := \inf_\alpha \sup_{\beta \geq \alpha} x_\beta$$
Show that $\liminf x_\alpha \leq \limsup x_\alpha$ and that $x_\alpha \to x$ iff
$$x = \liminf x_\alpha = \limsup x_\alpha$$
19. Let $\mathcal{I}$ be a directed set, and let $(\Omega, \mathcal{T})$ be a topological space that is also a vector space (DO I NEED it to be a TVS?). The set of all nets mapping from $\mathcal{I}$ to Ω is a vector space (see Section REF). Show that *limit* is a linear operator on this space. Assuming $(\Omega, \mathcal{T})$ is a complete lattice, show that *liminf*, and *limsup* are linear operators as well.

3 Bases

A is a collection of subsets $\mathbb{B}$ is called a base (or *synthetic basis*) if

- Its sets cover the space Ω. (SIDENOTE: A collection of sets $\mathbb{S}$ is said to cover (or to "be a *cover* for") a set X if every $x \in X$ is in the union over all $S \in \mathbb{S}$.)
- Any pair-wise intersection from $\mathbb{B}$ can be expressed as a union of sets in $\mathbb{B}$.

While any collection of sets generates a topology, a base has a particularly nice relationship to the topology it generates, as we will see.

20. Is every topology generated by some base?

Notice that the above definition makes no reference to a topology on the space. The property of being a base is *intrinsic* to $\mathbb{B}$ and not relative to any topology. On the other hand, we say that a collection $\mathbb{B} \subseteq \mathbb{T}$ is a basis (or *analytic basis*) for the topology $\mathbb{T}$ if every set in $\mathbb{T}$ can be expressed as a union of sets from $\mathbb{B}$. Crucially, the two concepts are related by the following fact.

Theorem 3.1. *$\mathbb{B}$ is a base iff $\mathbb{B}$ is a basis for $\tau(\mathbb{B})$.*

21. Prove Theorem 3.1.

Above, we've defined a number of aspects of topological spaces in terms of *neighborhoods*. The following fact ties neighborhoods to basis sets.

Corollary 3.2. *Let B be a basis for $\mathbb{T}$. Then M is a neighborhood of x iff it contains a $B \in \mathbb{B}$ that contains x.*

A collection $\mathbb{B}_x \subseteq \mathbb{T}$ is called a local basis (with respect to $\mathbb{T}$) at x if for every open set T containing x, there is some $B \in \mathbb{B}_x$ that is a subset of T. (SIDENOTE: Trivially, the topology $\mathbb{T}$ is itself a local basis at every point.) It's easy to observe an analogue of Corollary 3.2 in this context: M is a neighborhood of x iff it contains a $B \in \mathbb{B}_x$ that contains x. The following theorem clarifies the relationship between the concepts of *basis* and *local basis*.

Theorem 3.3. *$\mathbb{B}$ is a basis iff it is a local basis at every point.*

22. Prove Theorem 3.3.

The *basis* concepts are central to tying topology to other fields of functional analysis.

4 Continuous functions

Throughout this subsection, let f be a mapping from one topological space to another. f is called an open map if the *image* of every open set in the domain is an open set in the codomain. The "reversed" condition turns out to be much more important: f is called continuous if the *pre-image* of every open set in the codomain is an open set in the domain. (SIDENOTE: This characterization may not seem meaningful, but it turns out to be equivalent to the usual definitions of continuity in familiar contexts.)

When checking that f is an open map, it is sufficient to check any base of the domain's topology. Likewise, when checking for continuity, it is sufficient to check any base of the codomain's topology.

A nice thing about continuous functions is that limit can "pass through" them: $\lim f(x_n) = f(\lim x_n)$.

Theorem 4.1. *If f is a continuous mapping from $(\Omega, \mathbb{T})$ to another topological space, then $x_n \to x$ implies $f(x_n) \to f(x)$.*

23. Prove Theorem 4.1.

Any f for which $\lim f(x_n) = f(\lim x_n)$ holds for all convergent sequences is called sequentially continuous. In general, a function can be *sequentially continuous* without also being *continuous*. Sequential spaces are topological spaces for which continuity and sequential continuity are equivalent. Every first-countable space is sequential.

sequential spaces - sequences determine the topology - what exactly does that mean? IN CERTAIN types of sequential spaces (Frechet-Urysohn spaces), the CONVERSE TO [limit of sequence implies limit point] is true as well. that is, - IF x is a limit point for S, then there exists a sequence in S that converges to x.

A continuous function from a separable space to a Hausdorff space is determined by its values on a dense subset. AS A RESULT, the set of real-valued functions on a separable space has cardinality no greater than $\aleph_1$. (why?)

If f maps from Ω to a topological space, the initial topology on Ω with respect to f is the *coarsest topology* on Ω such that f is continuous. It is exactly the topology generated by the pre-images of all the open sets in the codomain. The initial topology with respect to a *collection of functions* is the coarsest topology for which all of them are continuous.

4.1 Homeomorphisms

topological space isomorphisms

Make reference to category theory here

5 Cataloguing topological spaces

5.1 Separation

Two points in Ω are topologically indistinguishable if they have the same neighborhood system (SIDENOTE: Otherwise, they are *topologically distinguishable*.); the topology is *blind* to any differences between such points. There are a number of important regularity conditions relating to how well a topology "separates" things.

Given a topological space $(\Omega, \mathcal{T})$ and a pair of items, each of which is either a point in Ω or a subset of Ω, we say that the two items are

- separated if each has a neighborhood that doesn't contain the other,
- separated by neighborhoods if they have a disjoint pair of neighborhoods.

Clearly *separation by neighborhoods* is a stronger condition than *separation*.

condition name	defining axiom
R_0 (symmetric)	any pair of topologically distinguishable points are s
R_1 (preregular)	any pair of topologically distinguishable points are s
R_2 (regular)	any closed set is separated by neighborhoods from a
R_3 (normal)	any two disjoint closed sets are separated by neighbo
R_4 (completely normal)	any two separated sets are separated by neighborho
R_5 (perfectly normal)	any closed set is a countable intersection of open set

It can be shown that these regularity conditions are (strictly) increasing in strength:

$$R_5 \Rightarrow R_4 \Rightarrow R_3 \Rightarrow R_2 \Rightarrow R_1 \Rightarrow R_0$$

They relate to how well a topology separates its *topologically distinguishable* points, but often we're interested in how well it separates *all* of its points.

A Kolmogorov space (or T_0*-space*) is a topological space in which *all points are topologically distinguishable*. In fact, we can construct a Kolmogorov space from any topological space using the fact that topological indistinguishability is an equivalence relation. By *condensing* the points into their equivalence classes and defining the neighborhood system of each equivalence class by the (common) neighborhood system of its original points, we get a Kolmogorov space, which is called the Kolmogorov quotient of the original topological space.

By adding axiom T_0 to any of the above regularity axioms, we get a space in which *all points* are separated in the specified ways. There is a convenient naming scheme for such spaces:

$$T_{k+1} := T_0 + R_k$$

A T_2-space (i.e. a preregular Kolmogorov space) is also called a Hausdorff space. In other words, in a Hausdorff space every pair of points is separated by neighborhoods. Because the R_k conditions increase in strength, so do the T_k conditions.

The defining conditions of the above spaces are among the separation axioms, a classification scheme for topological spaces. This detailed cataloguing allows us to keep track of exactly how much separation structure is necessary and/or sufficient for various other topological properties that we'll study.

24. Show that a topological space is Hausdorff iff every net converges to at most one point.
25. Does the result of Exercise 24 still hold if *net* is replaced with *sequence*?

In a Hausdorff space, each point is the unique intersection of its neighborhood basis. (or neighborhood system?) The neighborhood basis is a directed set because it is closed under intersection. - but this is supposed to index points... do we take one point from each neighborhood?

does preservation of net convergence provide an equivalent definition of continuous functions?

5.2 Size

There are a number of ways of quantifying the size of a topological space. Some of the relationships between these notions of size will be explored in the exercises.

5.3 Separability

Does the cardinality of a dense subset constrain the cardinality of Ω? In general, no. Consider the topological space comprising some set Ω and its trivial topology $\{\varnothing, \Omega\}$. Any singleton of an element in Ω is a dense subset. So regardless of the size or nature of Ω, *there is a topology* for which Ω has a dense subset of cardinality at most 1. However, for Hausdorff spaces, dense subsets do tell us something about the cardinality.

1.1* Suppose a Hausdorff space has a dense subset of cardinality a. Show that the cardinality of the full space is no greater than 2^{2^a}.

A set in a topological space is separable if it contains a *countable* dense subset. In Section 5.1, we defined *separated* and *separated by neighborhoods*. *Separablility* is a *different concept*, despite the similarity of the terms. The topological space itself is called *separable* if the full set is separable.

26. Is the union of countably many separable subsets also separable?
27. Show that every *open* subset of a separable space is separable. However, its not the case that *every subset* of a separable space is necessarily separable Devise an example to prove this.
28. Suppose f is a continuous function with a separable domain. Show that its range is also separable.

COMBINE this with a compactness result - Show that separability is preserved by continuous mappings. Show also that compactness is preserved by continuous mappings.

The convenience of separability will become increasingly apparent as we continue our study of functional analysis.

5.4 Compactness

A number of important classifications of the *size* of a set are related to the properties of its open covers. An *open cover* is a cover whose sets are all open. A set in a topological space is called

- Lindelöf if every open cover of the set contains a *countable* subcover,
- compact if every open cover of the set contains a *finite* subcover,
- σ-compact if the set has a countable partition into compact subsets.

It is immediate that the conditions here are listed from strongest to weakest. Similar classifications are related to the ability of its nets to "escape." A set in a topological space is called

- net compact if every net in the set has a convergent subnet,
- sequentially compact if every sequence in the set has a convergent subsequence.

These terms are applied to the topological space itself if the full set satisfies the given condition.

EXERCISE: Does *net compact* imply *sequentially compact*? NO - given a sequence, its subsequences are all subnets, but its subnets aren't necessarily subsequences! example at link:

Proposition 5.1. *Let $(\Omega, \mathcal{T})$ be a Hausdorff space. Then a subset is compact iff it is net compact.*

See Theorem 5.2 for a sufficient condition for the equivalence of compactness and sequential compactness.

MOVE THE FOLLOWING two facts TO metric spaces document.

Theorem 5.2 (The Bolzano-Weierstrass Theorem)**.** *In any metric space, a subset is compact iff it is sequentially compact.*

Theorem 5.3 (The Heine-Borel Theorem)**.** *In any metric space, a subset is compact iff it is both complete and totally bounded.*

Show that a net $\{x_\alpha\}$ in a metric space converges to x iff $d(x_\alpha, x) \to 0$. (INTERPRET $d(x_\alpha, x)$ in terms of a net as well - this is the right way of thinking about continuous limits of real numbers, i think - nets with the usual ordering on the reals, making them a directed set.)

Suppose f is a continuous function with a compact domain. Show that its range is also compact.

A set is called relatively compact if its closure is compact. - make up an exercise so that i can introduce this term.

A nice property of compactness is that it is preserved by continuous mappings. If S is compact and f is a continuous function, then the image $f(S)$ is a compact subset of the codomain. In particular, if f is real-valued has has a compact domain, then its range is a compact subset of $\mathbb{R}$. It can be shown that every compact subset of $\mathbb{R}$ has a greatest and a least element, so f attains a maximum and minimum. (SIDENOTE: This is a more general statement of the *Extreme Value Theorem*.)

MAKING a topological space into a compact space: "The methods of compactification are various, but each is a way of controlling points from 'going off to infinity" by in some way adding "points at infinity" or preventing such an "escape'."
Is this related to "compactly-generated spaces"?

also do

σ-compact

and locally compact - although this one is about the local properties of a space rather than the spaces overall size.

First and second countability

Recall from Section REF, that the *span* of a set of vectors is the "smallest" vector space containing them all; it is exactly the set of their linear combinations. Typically there are typically (infinitely) many subsets of vectors that span the whole space. We characterize the size of a vector space by its dimension, the cardinality of the smallest spanning sets. Similar reasoning is useful for topological spaces. However, extending a collection to its "smallest" topology is not as straightforward except for special types of collections.

Basis concepts provide some useful ways of characterizing the sizes of topological spaces. A topological space is first-countable if every point has a *countable* local basis.

1.2* A consequence of Exercise REF is that if a set is closed it must contain all its sequences' limits. One might wonder whether a converse is true: *if a set contains all its sequences' limits, then it is closed.* This converse is actually not true in general. Prove that it *is* true for first-countable spaces. (Thus, any set in a first-countable space is closed iff it contains all its sequences' limits.)

1.3 In Exercise REF, we saw that convergent sequences in a Hausdorff spaces must have unique limits. For first-countable spaces, show that the converse is true as well. (Thus, if a topological space is first-countable, then it is Hausdorff iff its convergent sequences have unique limits.)

1.4 Suppose a first-countable Hausdorff space has a dense subset of cardinality a. Show that the cardinality of the full space is no greater than 2^a. Compare this to Exercise **1.1**.

A topological space is second-countable if its topology has a countable basis. Because the basis is a local basis at every point, *second-countable implies first-countable.*

1.5 Devise an example of a topological space that is first-countable but not second-countable.

1.6 Prove that every second-countable space is separable.

A collection of subsets is locally finite if every point has some neighborhood that intersects only finitely many of the collection's sets. A collection is called countably locally finite if it is a union of countably many locally finite collections. The importance of a *basis* being countably locally finite will become clear in Theorem REF.

IN PAVINGS section make sure to DEFINE a countably-generated space - make it clear that this concept applies to all these "generated" spaces.

EXERCISE: A topological space is second countable iff it is countably generated - second countable implies countably generated is automatic - for the other direction, take any countable collection. the topology it generates is the same as the topology generated by itself along with its pairwise intersections, which are together a countable collection as well.

6 Topological completeness